SCHEMES AND VARIETIES

CHIRAG LAKHANI
NORTH CAROLINA STATE UNIVERSITY

1. Introduction

In this paper we seek to understand the general notions of a variety and scheme. A scheme can be thought of a generalization of a variety. We will first describe the notion of a variety and the type of spaces in which they live. The notion of a variety is inherently a geometric structure but it can be defined as an ring. This ring-like notion of a variety gives us the natural setting in which we will look at schemes. The basic ideas of mappings, topologies, and functions on varieties and schemes will be discussed.

2. Varieties

2.1. **Affine Varieties.**

First we define a natural space in which the central objects of study, varieties, are found. This space is called the affine n-space.

Definition 1 (Affine N-Space)**.** Given an algebraically closed field k the **affine n-space** is defined as the an n-tuple of elements of k. E.g. p=(a_1, ..., a_n) where each $a_i \in k$, we will denote such a space A^n.

The most common example of an affine space occurs on the algebraically closed field $\mathbb{C}$. In this space p =(k_1, ..., k_n) where each $k_i \in \mathbb{C}$. This space is aso denoted $\mathbb{C}^n$.

This gives us the proper space in which we can define our notions of algebraic sets and varieties. In this paper we will denote our algebraically closed field as k, but it can be thought of as $\mathbb{C}$ in order to visualize geometrically.

Definition 2 (Algebraic Set)**.** An **algebraic set** on A^n is defined as the common zeroes of a collection of polynomaials in k[x_1,...,x_n].

Examples 3. A clear example of this comes in the form the common zeroes of the complex functions f(x,y) = x^2+y^2-1 and g(x,y) = y-x^2. These two curves are defined on the affine n-space A^2 also known as $\mathbb{C}^2$ in this case. The first function is the complex sphere and the second function is the parabola because we get the standard equations when we set f(x,y) and g(x,y) equal to zero.

Now for such geometric objects we hope to have some sort of topology for such a space. There is a corresponding topology called the Zariski topology which will be defined on this space.

Definition 4 (Zariski Topology)**.** A closed set in **Zariski topology** of A^n is defined as an algebraic set defined on A^n.

The open sets of the Zariski topology are therefore defined as complements of algebraic sets on A^n. We leave it as an excercise to verify that this topology satisfies the topology axioms *(Hint: it is easiest to verify the closed set axioms).* So given an algebraic set on A^n we can define its Zariski topology as the subspace topology on A^n.

An algebraic variety is simply a restricted case of the algebraic set. Given the algebraic set h(x,y)=x^2+y^2 = (x+iy)(x-iy), one can notice that this polynomial is reducible. Geometrically, this means that this algebraic set is really the union of two algebraic sets (x+iy) and (x-iy). It's easy to see this because the zeros of the polynomial occur when x=-iy or x=iy hence h(x,y) is the union of these two lines. This motivates our definition of an algebraic variety.

Definition 5 (Affine Algebraic Variety)**.** An **affine algebraic variety** on A^n is defined as an irreducible algebraic set (closed subset) of A^n.

This definition motivates the correspondence between algebraic varieties and rings. A variety X on A^n has an associated ideal I(X) = (f ∈ k[x_1,....,x_n] | f(p)=0 ∀p∈ X) in the ring k[x_1,....,x_n]. Now it is easy to check that this set I(X) is an ideal in this ring. If we add any two polynomials which vanish on all points of X then their sum will also vanish on X. Similarly if we multiply a polynomial which vanishes on X by any other polynomial in k[x_1,....,x_n] then its product will also vanish on X.

Definition 6 (Affine Coordinate Ring)**.** The **affine coordinate ring** of X is $k[x_1, ..., x_n]/I(X)$. This can also be stated as the polynomial ring k[x_1,....,x_n] modulo the ideal I(X).

Examples 7. In the example of the complex sphere x^2+y^2=1 we see that I(X) will only be generated by the polynomial f(x,y) = x^2+y^2-1. So the affine coordinate ring will be k[x,y]/(x^2+y^2-1).

In the case of the parabola the ideal of the parabola is generated by (y-x^2). So the affine coordinate ring is k[x,y]/(y-x^2)≅ k[x].

Remark 8. It is important to note that in these two examples there is a unique generator for the ideal of the variety. There are many instances in which the corresponding ideal does not have a unique generator nor can be written as a principal ideal.

A key theorem that relates algebraic varieties and affine coordinate rings is called Hilbert's Nullstellensatz.

Theorem 9 (Hilbert's Nullstellensatz)**.** *Given an algebraically closed field k and I is an ideal in k[*x_1*,....,*x_n*] and a polynomial f∈k[*x_1*,....,*x_n*]. Now if all points of* A^n *that vanish on all of the polynomials in I also vanish on f then* f^r ∈ *I for some r≥ 1.*

Corollary 10. *There is a bijective correspondance between affine varieties in* A^n *and radical ideals of k[*x_1*,....,*x_n*].*

Remark 11. A radical ideal, I, of a ring, R, is simply an ideal where if f^r ∈ I then f ∈ I.

This gives us the unique correspondance between algebraic varieties and coordinate rings that we wanted.

2.2. **Projective Varieties.** A natural way to extend affine varieties is by embedding them into a projective space.

Definition 12 (Projective N-Space)**.** **projective n-space**,$\mathbb{P}^n$, is the set of one dimensional subspaces of the the affine space A^{n+1}.

Projective n-space is a space where points that differ by only multiple by k are identified in A^{n+1}, $\mathbb{P}^n \sim A^{n+1}$ / ([x_0,...,x_n]$\sim$ [λx_0,...,λx_n] $\lambda \in k$). Note: [x_0,...,x_n] has to be non-zero in $\mathbb{P}^n$

Definition 13 (Projective Variety)**.** A **projective variety** in $\mathbb{P}^n$ is an irreducible algebraic set in $\mathbb{P}^n$

Remark 14. It is best to think of a projective variety in terms of its homogeneous polynomials. A homogeneous polynomial is simply a polynomial where all of the monomials have the same degree. So in k[x_0,....,x_n] a homogeneous polynomial of degree d will have f(λx_0,...,λx_n)= λ^d f(x_0,...,x_n). We can easily see that any two points in A^{n+1} which differ by a scalar multiple λ will vanish on the same homogeneous polynomial. So the ideal associated to a projective variety will be generated by such homogeneous polynomials. We denote the projective ideal associated to the projective variety X as $I_p(X)$. We can describe the associated coordinate ring to this projective variety as k[x_0,....,x_n] / $I_p(X)$.

Examples 15. Returning to our example of of the complex sphere x^2+y^2=1 and parabola x^2=y we can projectivize this sphere using homogenization. Homogenization takes a polynomial of n-variables and multiplies each monomial element by n+1 variable so that all of the monomial have the same degree. In the case of both complex sphere and parabola we multiply by a third variable z. This projective sphere is written as x^2+y^2=z^2 and the projective parabola is x^2=yz. The ideal of the projective sphere is now generated by the homogeneous function f(x,y,z)=x^2+y^2-z^2. The the homogeneous ideal of the projective parabola is g(x,y,z)=x^2-yz.

Projective varieties can be thought of as covered by affine varieties. In $\mathbb{P}^n$ every p=[x_0,...,x_n]$\sim$ [λx_0,...,λx_n]. Since p is non-zero there is a non-zero x_i, so we set $\lambda = \frac{1}{x_i}$. So p=[$\frac{x_0}{x_i}, .., 1, .., \frac{x_n}{x_i}$]$\cong$ [$\frac{x_0}{x_i}, .., \frac{x_n}{x_i}$] which is in A^n. So all points in $\mathbb{P}^n$ can be projected in this manner to some affine space A^n. Analogously we can project any projective variety in $\mathbb{P}^n$ to a variety in A^n `(See Hartshorne p.10)`.

2.3. **Morphisms.** Now that we have seen the fundamental ideas regarding varieties a natural step is to understand mappings between these objects. We first define a regular function on a variety.

Definition 16 (Regular Function)**.** Given an affine variety X on A^n and a function f: X $\rightarrow$ k, the function f is regular at a point p$\in$ X if there is an open set U $\subseteq$ X containing p where f=$\frac{g}{h}$ for g,h $\in$ k[x_1,....,x_n] and h is non-zero on U.

The given definition for a regular function is regular at a point, but we can say that a function is regular on an open subset U $\subseteq$ X if it is regular at every point on U. Similarly a regular function is regular on X if it is regular at every point on X.

In the case of a projective variety Y $\subseteq \mathbb{P}^n$, f is regular if we restrict the polynomials to homogeneous polynomials $f_{\mathbb{P}}, g_{\mathbb{P}} \in k[x_0, ..., x_n]$ where degree $f_{\mathbb{P}} - g_{\mathbb{P}} = 0$.

For a given variety X (affine or projective) the collection of functions which are regular on X has a ring structure, we will represent this as θ(X). It is left as an excercise to see that if we add or multiply any two regular functions on X then we will get another regular function. The collection functions regular at a point p $\in X$ will form the ring of germs of a function. The ring of germs is a pair (U,f) where U is the open set about p where f is regular. Any two pairs (U,f) and (V,g) are equivalent if there is an open set W $\subseteq$ U$\cap$ V where f=g, we denote this ring θ_{X_p}.

It can be seen that these collections regular functions are isomorphic the coordinate ring.

Theorem 17. *Given an affine variety X on A^n with corresponding affine k[x_1,....,x_n]/I(X) we have:*
a) $\theta(X) \cong k[x_1, ..., x_n]/I(X)$
b) $\theta(X)_p \cong k[x_1, ..., x_n]/I(X)_{m_p}$ $\leftarrow$ *this is the affine coordinate ring localized at the maximal ideal m_p corresponding to the point p.*

Given an affine variety Y on $\mathbb{P}^n$ with homogeneous coordinate ring k[x_0,....,x_n]/J(X) (remember J(X) is a homogeneous ideal of the ring) we have:
a) $\theta(Y) \cong k$
b) $\theta(Y)_p \cong k[x_0, ..., x_n]/J(Y)_{m_p}$ $\leftarrow$ *this is the homogeneous coordinate ring localized at the homogeneous maximal ideal m_p corresponding to the point p.*
(See Hartshorne p. 16-18 for further details and proof)

Definition 18 (Morphism)**.** Given two varieties X and Y a continuos map F : X $\rightarrow$ Y with respect to the Zariski topology where given any open subset U $\subseteq$ Y and any regular function f on U, f $\circ$ $F : F^{-1}(U) \rightarrow k$ is a regular function.

So a morphism between two varieties will be continuous and will pull back regular functions. The bijective correspondance between varities and radical ideals gives a unique correspondance between morphisms between varieties and maps between their coordinate rings. In fact their is a natural bijective correspondance between the set of morphisms F : X $\rightarrow$ Y and the set of mappings of coordinate rings G: k[x_1,....,x_n]/I(Y) $\rightarrow$ $\theta(X)$ `(See Hartshorne p.19 and 20)`

3. Schemes

3.1. **Sheaves.** We make a brief interlude into sheaf theory before we define a scheme. Sheafs are very useful objects in mathematics. They arise not only algebraic geometry but also differential geometry, number theory, commutative algebra, algebraic topology, and complex analysis just to name a few branches. We take a functorial approach to the definition of sheaves but we will show how it connects another definition called espace etale.

Definition 19 (Presheaf)**.** Given a topological space X a presheaf F of abelian groups is a functor from the topological space X to abelian groups which satisfy the following properties:
1) For every open set V $\subseteq$ X F(V) is an abelian group 2) For every U $\subseteq$ V $\subseteq$ X there is a group homormorphism ρ_{VU} : F(V) $\rightarrow$ F(U) which satisfy the following properties:

a) F(∅) = 0, where ∅ is the empty set.
b) ρ_{UU} : F(U) $\rightarrow$ F(U) is the identity map
c) The maps preserve composition i.e. for open sets U $\subseteq$ V $\subseteq$ W $\subseteq$ X,

$\rho_{VU} \circ \rho_{WV} = \rho_{WU}$.
(Note: an element of F(U) is called a section and the map ρ_{VU} can be thought of as the restriction onto U. For an section s $\in$ F(V) we denote the restriction onto U as $s|_U$)

Definition 20 (Sheaf)**.** A presheaf becomes a sheaf if they satisfy these two additional conditions:
1) Given an open set U and a open covering (U_i) of U, if there is a section s $\in$ F(U) where $s|_{U_i}$=0 for each of the open covers U_i then s=0.
2) For such an open covering (U_i) of U if there is an element $s_i \in F(U_i)$ where on the intersections $U_i \cap U_j$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there is a section s $\in$ F(U) where $s|_{U_i} = s_i$.

The sheaf axioms are very useful because they allow local data on an open covering to be defined globally. Once we are given a sheaf we can try to look at the sheaf at a point using a stalk. Here is the formal definition of a stalk.

Definition 21 (Stalk of a sheaf)**.** Given a sheaf F on the topological space X. If p $\in$ X then the stalk F_p is the direct limit of groups F(U) as U ranges over all open sets containing x.

This definition, while very precise, can be hard to penetrate. Intuitively the sheaf of the stalk relates back to the discussion regarding germs of functions. Elements in F_p are germs of sections of the sheaf F. This means that an element in F_p can be representated by (U,s) where U $\subseteq$ X is an open subset and s is a section of F(U). Two elements (U,s) and (V,t) are equal if there is an open subset W $\subseteq V \cap U$ where $s|_W = t|_W$.

A subsheaf G of a sheaf F abelian groups is defined to be be subgroups of the sheaf F. So for every open set U of X G(U) is a subgroup of F(U). Sheaves are not restricted to only abelian groups there can be sheaves of rings, modules, and sets as well.
Mappings between sheaves is very important in the context of schemes. Given two sheaves F and G on the topological space X, the map between these sheaves is simply a collection of maps $\phi_U : F(U) \rightarrow G(U)$ for all open subsets U $\subseteq$ X. Given a continuous map f: X $\rightarrow$ Y of two topological spaces X and Y, we can define a direct image sheaf and inverse image sheaf. For a sheaf F on the space X the direct image sheaf is defined as $(f_* F)(V) = F(f^{-1}(V))$ for any open subset $V \subseteq Y$. So if we have a sheaf defined on the topological space X we can push it forward to become a sheaf on Y. The direct image sheaf will be important in the construction of schemes. The discussion was restricted to sheaf of abelian groups but the same results follow if we have a sheaf of rings, sheaf of sets, or sheaf of modules.

Examples 22. As we stated before sheaves are important not only in algebraic geometry but also differential geometry. Given a manifold M (M can be real of complex) the differential forms on M form a ring structure. It can be easily proven that they form a sheaf on the manifold M. The partitions of unity on a manifold M are important when trying to glue together differential forms defined on open covers of a manifold.
Going back to our discussion of varieties, the ring of regular functions of a variety X will be a sheaf on the variety as well. The discussion about germs of regular

functions at a point p ∈ X is equivalent to stalks of a sheaf of rings on the variety X.

3.2. **Schemes.** Having defined sheaves and given some of the properties associated to sheaves we are now ready to define a scheme. A scheme can be thought of as a generalization of a variety. It is very useful because it's versatility. Schemes come up as number rings in algebraic number theory to compact Riemann surfaces in differential geometry. Before defining a scheme we need to look at a topological space called Spec.

Definition 23 (Spec)**.** Given a ring A we define Spec(A) to be the set of all prime ideals of A. The closed sets of Spec(A) are sets of prime ideals that contain some ideal in A, which we will denote V(I). So the open sets of Spec(A) are of the form Spec(A)-V(I).

It can be seen that given two closed sets V(I) and V(J), where I and J are ideals in the ring A, V(I)∪V(J) = V(E), where E is the ideal generated by the intersection of I and J. Similarly V(I) ∩ V(J) = V(I∪J), this can be generalized to an arbitrary number of intersections so that it satisfies the closed set axioms of a topological space. On this topological space it becomes neccesary to define a sheaf structure. Spec(A) has a collection of open sets called the the basic open sets. An element f ∈ A can generate a principal ideal (f), now the basic open sets D(f):=Spec(A)-V((f)). Intuitively it can be seen that prime ideals (f) are the smallest type of ideals in a ring A so V((f)) will be a very large collection of prime ideal therefore D(f) will be a small collection ideals. These sets D(f) basically form a base for the topology of Spec(A) e.g. any open set U is the union of sets of the form D(f).

Before we define the sheaf of rings on Spec(A) we need to comment on the localization of a ring. If we are given a ring A localization means that we are adding multiplicative inverses by extending the ring into fractions. If we are given a ring A and a prime ideal p on A we can define a localization of A at p $A_p{:=}\frac{A}{A-p}$. So the fractioned ring will have elements of the ring A in the numerator and elements of the ring A which are not in p in the denominator. The localized ring will still have a ring structure. If a,c ∈ A and b,d ∈ (A-p) then $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$, where ad+bc ∈ A and bd ∈ (A-p). Similarly if a,c ∈ A and b,d ∈ (A-p) then $\frac{a}{b}*\frac{c}{d}=\frac{ac}{bd}$, where ac ∈ A and bd ∈ (A-p). We are now ready to define the sheaf of rings. If we have an element f ∈ A we can define localizatin at an element as $A_f{:=}\frac{A}{1\cup f^n n\geq 0}$ which means that elements of A_f are of the form $\frac{s}{1}$ or $\frac{s}{f^n}$ for s ∈ A

Given the topological space Spec (A), which consists of prime ideals of A, we can localize A at each prime ideal p ∈ spec(A). So on open sets U ⊆ Spec(A) the sheaf of rings $\mathcal{O}$(U) is defined as the set of functions s:U → $\coprod_{p\in U} A_p$, which has a ring structure.

This sheaf structure is very important because it can be proven that the sheaf structure is actually related to ring.

Theorem 24. *Given the ring A with associated space Spec(A) and sheaf $\mathcal{O}$ as defined above, we have:*
a) The stalk $\mathcal{O}_p \cong A_p$ for $p \in Spec(A)$
b) Given an element $f \in A$, $\mathcal{O}(D(f)) \cong A_f$

c) $\mathcal{O}$ Spec(A)) $\cong A$
(See Hartshorne p.71 for further details and proof)

If we recall from the sheaf examples we stated that the ring of regular functions on a variety X can be defined in terms of a sheaf. In the discussion of regular functions we stated isomorphisms between the ring of regular functions and the coordinate ring associated to the variety. In the previous theorem we get analogous results, but we will see more clearly the correspondance later.

Spec(A) is the fundamental building blocks in studying schemes. The previous theorem states that the stalk of a sheaf of Spec(A) has a local ring structure. When we are given a topological space X with an associated sheaf of rings θ_X when we look at the stalk θ_{X_p} at a point p $\in$ X we want it to be a local ring as well.

Definition 25 (Locally Ringed Space)**.** Given a topological space X with an associated sheaf of rings θ_X, this space is a locally ringed space if the stalks θ_{X_p} are local rings.

Remark 26. Based on theorem 24 it is quite obvious that Spec(A) is a locally ringed space.

A morphism between two locally ringed spaces, (X, θ_X) and (Y, θ_Y) consists of a pair, (f,f'), of maps. The map f is the continuous map between the topological spaces, f : X $\to$ Y. The map f' is a sheaf map between θ_Y and the direct image sheaf $f_*\theta_X$. In terms of open sets U $\subseteq$ Y f': θ_Y (U) $\to$ θ_X $(f^{-1}(U))$. So f' is a sheaf map on Y.

An isomorphism of locally ringed spaces must be a homeomorphism on topological spaces and an isomorphism as a sheaf map.

Before we define a scheme in its full generality we can define an affine scheme.

Definition 27 (Affine Scheme)**.** A locally ringed space (X, θ_X) is an affine scheme if it is isomorphic (Spec(A), $\mathcal{O}_X$).

Examples 28. In the case of the affine line A^1 the affine coordinate ring will be k[x]. Now Spec(k[x]) consists of prime ideal of the polynomial ring k[x]. For any point p $\in$ A^1 there is a corresponding prime ideal (x-p). Since (x-p) is a maximal ideal in k[x] then in terms of the Zariski topology V((x-p)) will only be the ideal (x-p). So the prime ideals (x-p) are closed points in the sense of the Zariski topology, which is equivalent to the closed points in the geometric sense of the affine line.

In the case of the complex sphere x^2+y^2=1 and complex parabola y-x^2 in A^2 the affine coordinate rings are k[x,y]/(x^2+y^2-1) and k[x,y]/(y-x^2) respectively. So if we take Spec(k[x,y]/(x^2+y^2-1)) or Spec(k[x,y]/(y-x^2)). Then we get the prime ideals associated to those rings. The points in each variety will correspond to maximal ideals in Spec but there can also be other prime ideals in Spec which can correspond to subvarieties of this variety.

A variety X is a locally ringed space because Theorem 17 shows that the stalks on the sheaf of regular functions on X are local rings.

If we are given an affine variety, by the Hilbert's Nullstellensatz, there is a corresponding affine coordinate ring. So the topology of this affine variety in terms of the Zariski topology will be homeomorphic to Spec of its affine coordinate ring with respect to its topolgoy. Now the sheaf of regular functions on the variety is isomorphic to the affine coordinate ring, just as the sheaf of rings on Spec is isomorphic to the affine coordinate ring. This gives us the isomorphism of locally ringed spaces between (X, θ_X) and (Spec($k[x_1, ..., x_n]/I(X)$), $\mathcal{O}_{k[x_1,....,x_n]/I(X)}$).

In the case of projective varieties we saw that it is covered by affine varieties. This makes it neccesary to generalize the idea of an affine scheme, which motivates the definition of a scheme.

Definition 29 (Scheme)**.** A locally ringed space (X, θ_X) is a scheme if at every point p $\in$ X has an open neighborhood U which is homeomorphic to Spec(A) for some ring A and the sheaf $\theta_X|_U$ is isomorphic as a sheaf to $\mathcal{O}_{Spec(A)}$.

The beauty of schemes comes is seen not only because it is a generalization of varieties, but it can also take other forms other than varieties. Certain types of rings in number theory, which have no geometrical analogue, fit beautifully into the theory of schemes. This allows one to use the powerful technicque in algebraic geometry in number theory.

References

[1] M. Atiyah, I.G. MacDonald, *Commutative Algebra*, Perseus Publishing, 1994.
[2] D. Eisenbud, J. Harris, *The Geometry of Schemes*, Springer, 2000.
[3] W. Fulton, *Algebraic Curves*, Addison-Wesley Pub Co., 1989.
[4] R. Hartshorne, *Algebraic Geometry*, Springer, 1993.
[5] I. Shafarevich, *Basic Algebraic Geometry*, Springer, 1995.