Prime License Plate Numbers

Ordinary license plates in North Carolina have 3 letters followed by (up to) 4 numbers: ABC-1234. So what is the probability that a license plate number is prime? If we assume all numbers between 1 and 9999 are equally likely then we can answer the question by knowing the number of primes between 1 and 9999. A quick trip to the Internet answers this: there are 1229 prime numbers less than 10,000. So the probability that a license plate number is prime is $1229/9999 \approx 0.1229 \approx 1/8$. This question was not very (mathematically) interesting because we didn't really use any mathematics to answer it; all we had to do was Google something.

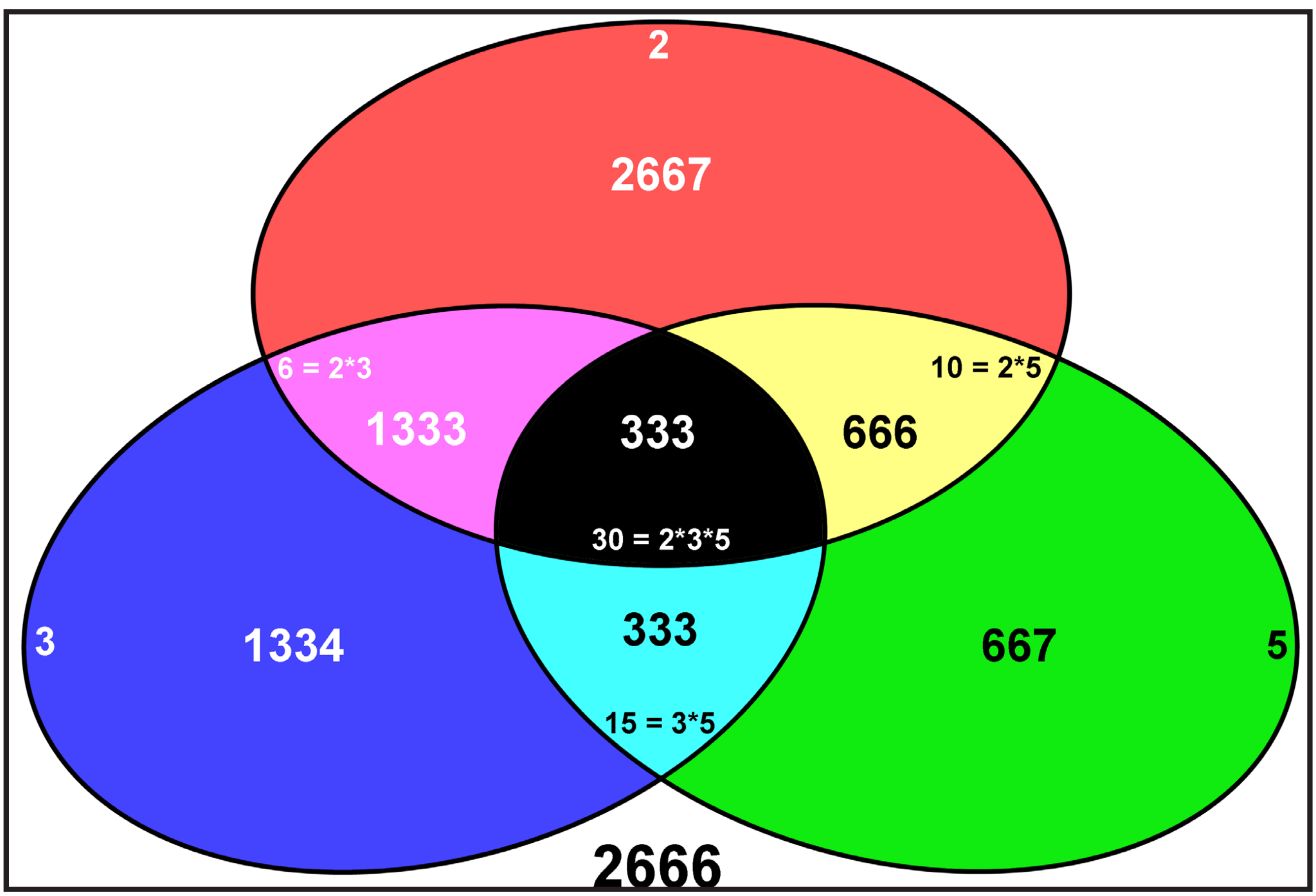
Some license plates are obviously *not* prime. We can easily eliminate those that are divisible by 2 (the last digit is even), by 3 (sum of the digits is a multiple of 3), or by 5 (the last digit is 0 or 5). Let's say that a number passes the (2-3-5)–test if it is not divisible by 2 and not divisible by 3 and not divisible by 5. So we have a more interesting question: If a license plate number passes the (2-3-5)–test, what's the probability that it's prime? Then the probability we're looking for is calculated by dividing the number of primes which pass the (2-3-5)–test by the number of all numbers which pass the test. [What are the prime numbers which fail the (2-3-5)–test?]

Counting the numbers which pass the (2-3-5)–test is a little tricky. They are most easily counted by counting the number of those which *fail* the test and subtracting that number from 9999. Even doing that is a little tricky. How can a number fail the test? It can be divisible by 2 or by 3 or by 5. These are easily counted:

divisible by 2: $9999/2 = 4999$ (discard remainder), # divisible by 3: $9999/3 = 3333$, and # divisible by 5: $9999/5 = 1999$.

If we add these three numbers, we get the wrong answer because some numbers have been counted more than once: 6 has been counted twice because it's divisible by 2 and by 3. And 30 has been counted three times! One simple way to get the correct count is to work with a Venn diagram of the situation (below). We fill it in from the inside out: the number of numbers divisible by 2 *and* by 3 *and* by 5 is $9999/30 = 333$. Then the number of numbers divisible by 2 *and* by 3 but *not* by 5 will be $9999/6 - 333 = 1666 - 333 = 1333$. Note that we calculate the total number divisible by 2 and by 3 and subtract off those which are also divisible by 5. We have filled in the whole Venn diagram accordingly. The number of numbers *failing* the (2-3-5)–test is the total of all the numbers in the diagram: 2667 + 1333 + 333 + 666 + 1334 + 333 + 667 = 7333. So the number passing the test is 2666. So we can finally answer the question: If a license number passes the (2-3-5)–test, then the probability that it is prime is:

(# primes passing (2-3-5)–test)/(# numbers passing (2-3-5)–test) = $1226/2666 \approx 0.4599$. So eliminating the obvious (those numbers divisible by 2 or by 3 or by 5) almost quadruples the likelihood the licence number is prime.



What if we consider those numbers which pass the (2-3-5-7)–test? The resulting Venn diagram is *much* more complicated and filling it in (without errors) is a major chore. We give it below (hopefully without errors). Adding all the numbers in the diagram and subtracting from 9999 gives 2285 as the number of numbers passing the (2-3-5-7)–test. So the probability that a license plate which passes the (2-3-5-7)–test is prime turns out to be

(# primes passing (2-3-5-7)–test)/(# numbers passing (2-3-5-7)–test) = $1225/2285 \approx 0.5361$, over ½!