Criteria for Divisibility by Small Primes
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Our context will be the set \( \mathbb{Z} \) of integers and, more particularly, the set of positive integers \( \mathbb{N} \ aka \) the set \{1, 2, 3, \ldots \}. One of the basic tools for working with integers is the Division Algorithm:

**Theorem.** For any integer \( n \) and any natural number \( d \) there are unique integers \( q \) and \( r \), with \( 0 \leq r < d \), so that \( n = q \cdot d + r. \)

Generally, \( d \) is referred to as the divisor, \( q \) the quotient, and \( r \) the remainder. If \( r = 0 \), so that \( n = q \cdot d \), we say that \( d \) divides \( n \). We write this in mathematical shorthand as \( d \mid n \). For example, \( 2 \mid 6 \) since \( 6 = 3 \cdot 2 \), \( 3 \mid 111 \) since \( 111 = 37 \cdot 3 \), and \( 5 \nmid 36 \) (5 does not divide 36) since \( 36 = 7 \cdot 5 + 1 \). In this case the unique remainder guaranteed by the Division Algorithm is 1, not 0.

**Easy Theorems:**
- If \( d \) divides any two of \( a \), \( b \), and \( a \pm b \), then \( d \) divides the third.
- If \( d \mid k \cdot a \) and \( d \) and \( k \) have no common factor (other than 1), then \( d \mid a \).

**Major Definition:** A natural number \( p > 1 \) is prime if \( a \mid p \) implies \( a = 1 \) or \( a = p \).

Mathematical notation is sometimes ambiguous. For example, if \( a \) and \( b \) are digits (integers between 0 and 9), what does \( ab \) mean? We frequently use it to mean \( a \) times \( b \) but it can also mean the concatenation of \( a \) and \( b \), i.e., our standard notation for the number 10\(a+b\). In the following we will (hopefully, consistently) use \( ab \) to denote the concatenation of \( a \) and \( b \); the product will be denoted \( a \cdot b \). Thus \( abcd \) is shorthand for \( a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d \cdot 10^0 \).

Here are some easy tests to determine if a number \( n = abcd \) is divisible by the prime numbers 2, 3, 5, 7, or 11. These tests work for any numbers; we use 4 digit numbers simply for the relative ease of exposition. A couple of these (2 and 5) you have known all your life:

**Divisibility by 2.** \( 2 \mid abcd \iff 2 \mid d, \ i.e., \iff d \) is even.

**Why?**
\[
d = abcd - abc0 = abcd - 10 \cdot abc \text{ so } 2 \notmid abcd \iff 2 \notmid d. \quad \blacksquare
\]

**Divisibility by 3.** \( 3 \mid abcd \iff 3 \mid (a + b + c + d) \).

**Why?**
Remember that \( abcd \) is shorthand for \( a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d \). So
\[
a + b + c + d = a \cdot (10 - 9)^3 + b \cdot (10 - 9)^2 + c \cdot (10 - 9)^1 + d
\]
\[
= a \cdot 10^3 - 3 \cdot a \cdot 10^2 \cdot 9^1 + 3 \cdot a \cdot 10^1 \cdot 9^2 - a \cdot 9^3
\]
\[
+ b \cdot 10^2 - 2 \cdot b \cdot 10^1 \cdot 9^1 + b \cdot 9^2
\]
\[
+ c \cdot 10^1 - c \cdot 9^1
\]
\[
+ d
\]
\[
= a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d + 9 \cdot (MESS)
\]
\[
= abcd + 9 \cdot (MESS)
\]

So we see that \( 3 \mid abcd \iff 3 \mid (a + b + c + d). \quad \blacksquare
\]

**Corollary: Divisibility by 9.** \( 9 \mid abcd \iff 9 \mid (a + b + c + d). \quad \blacksquare
\]

**Divisibility by 5.** \( 5 \mid abcd \iff 5 \mid d, \ i.e., \iff d = 0 \) or \( d = 5 \).

**Why?**
\[
d = abcd - abc0 = abcd - 10 \cdot abc \text{ so } 5 \notmid abcd \iff 5 \notmid d. \quad \blacksquare
\]

**Divisibility by 7.** \( 7 \mid abcd \iff 7 \mid (abc - 2 \cdot d) \).

**Why?**
\[
10 \cdot (abc - 2 \cdot d) = abcd0 - 20 \cdot d
\]
\[
= abcd0 + d - 21 \cdot d
\]
\[
= abcd - 21 \cdot d
\]

So we see that \( 7 \mid abcd \iff 7 \mid (abc - 2 \cdot d). \quad \blacksquare
\]
Divisibility by 11. $11 | abcd \iff 11 | (a - b + c - d)$.

Why?
This argument is very similar to that for divisibility by 3:

\[ a - b + c - d = -(a \cdot (10 - 11)^3 + b \cdot (10 - 11)^2 + c \cdot (10 - 11)^1 + d) \]
\[ = -(a \cdot 10^3 - 3 \cdot a \cdot 10^2 \cdot 11^1 + 3 \cdot a \cdot 10^1 \cdot 11^2 - a \cdot 11^3) \]
\[ - (b \cdot 10^2 - 2 \cdot b \cdot 10^1 \cdot 11^1 + b \cdot 11^2) \]
\[ - (c \cdot 10^1 - c \cdot 11^1) \]
\[ - d \]
\[ = -(a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d) + 11 \cdot (MESS) \]
\[ = -abcd + 11 \cdot (MESS) \]

So we see that $11 | abcd \iff 11 | (a - b + c - d)$. □

Corollary: $11 | abba, 11 | abccba, \text{ etc.} \, (\text{palindromic numbers with an even number of digits}).$

The elegant technique of determining divisibility by 7 is due to Stuart Savory and can be found at http://home.egge.net/.savory/maths1.htm. His very clever method can provide other prime divisibility tests. Here’s how it would work for $p = 13$.

Find the smallest multiple of 13 ending in 9 or in 1. In this case $3 \cdot 13 = 39$, so we are in the “9 case”. The multiplier will then be 4 (the leading digit of $39 + 1$) and we add instead of subtract: $13 | abcd \iff 13 | (abc + 4 \cdot d)$.

Here’s why:

\[ 10 \cdot (abc + 4 \cdot d) = abc0 + 40 \cdot d \]
\[ = abc0 + d + 39 \cdot d \]
\[ = abcd + 39 \cdot d \]

So we see that $13 | abcd \iff 13 | (abc + 4 \cdot d)$. □

Can you see why $17 | abcd \iff 17 | (abc - 5 \cdot d)$ and $19 | abcd \iff 19 | (abc + 2 \cdot d)$?

The technique will not work for 2 or 5 (WHY??) and it produces somewhat different looking (but basically the same) tests for 3 and 11 than the ones above:

- $3 | abcd \iff 3 | (abc + d)$
- $11 | abcd \iff 11 | (abc - d)$

Note. Savory’s tests are meant to be used recursively, that is, over and over on a sequence of smaller and smaller numbers until the final divisibility question is obvious. For example,

\[ 7 | 8911 \iff 7 | (891 - 2 \cdot 1) \iff 7 | 889 \iff 7 | (88 - 2 \cdot 9) \iff 7 | 70. \, \text{So} \, 7 | 8911. \, \text{(}8911 = 7 \cdot 1273).