Eudoxus Meets Cayley

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1. INTRODUCTION. The uses of geometry stretch back to the dawn of human history. The earliest written records (from Egypt and Babylonia) contain geometric observations, problems, and solutions. During the classical Greek civilization in the period 600–300 BCE geometry was organized and systemized into what we view today as formal mathematics, culminating in Euclid's *Elements*. By this time mathematics had encountered and resolved its first crisis, as identified by Eves [**1**, pp. 15–16]: the discovery of irrational numbers (in the fifth century BCE) and the consequent invalidation of all proofs that depended on the assumption that any two lengths were commensurable. When Euclid wrote the *Elements* he was able to incorporate (as Book 5) Eudoxus's brilliant resolution of the crisis through the development of proportional lengths and similar triangles.

It would be difficult to overstate the impact that this first textbook of axiomatic mathematics had on Western civilization. Everyone with a formal education learned Euclid. It is clear from its organization and presentation that Thomas Jefferson and the other writers of The Declaration of Independence knew the *Elements*. Abraham Lincoln, in the autobiographical sketch he wrote for the Chicago *Press & Tribune* when he ran for President in 1860, stated that "He studied and nearly mastered the six books of Euclid, since he was a member of Congress." Archimedes, Newton, Gauss, and the other giants of mathematics were masters of geometry. Thus it is with some trepidation that we present this article.

We begin with a few elementary results from classical geometry that Eudoxus and Euclid could certainly have understood. These results naturally lead to a question that is also easily understood in the framework of classical geometry. It is in searching for the answer to this question that we are led into the mathematics of Cayley.

2. TRIANGLE AND SUBTRIANGLES. Conventionally, the triangle $\triangle ABC$ is the union of its sides: the noncollinear segments between A and B, B and C, and C and A, respectively. A *chord* of $\triangle ABC$ is any segment whose end points are interior points of different sides. We call the point X on the segment between A and B for which the ratio $AX\!:\!XB = p\!:\!q$, where p and q are natural numbers, *the* $(p\!:\!q)$*-point from A to B*. Note the asymmetry of this definition: the $(p\!:\!q)$-point from A to B is the $(q:p)$-point from B to A. If X is the $(p\!:\!q)$-point from A to B and Y is the $(p\!:\!q)$-point from A to C, then the segment $\overline{XY}$ is called *the* $(p\!:\!q)$*-chord from A in* $\triangle ABC$. It is an easy exercise in similarity arguments to prove the following.

Theorem 2.1. *The* $(2\!:\!1)$*-chords from A, from B, and from C in* $\triangle ABC$ *all have the centroid of* $\triangle ABC$ *as midpoint. Consequently, these three chords are concurrent (share a common point).*

We are interested in properties of subtriangles, so to facilitate the development we adopt the following terminology and notation.

Definition 2.1. In $\triangle A_0B_0C_0$ let A_1, B_1, and C_1 be the $(p\!:\!q)$-points from A_0 to B_0, from B_0 to C_0, and from C_0 to A_0, respectively. Triangle $\triangle A_1B_1C_1$ is called *the*

$(p\!:\!q)$*-subtriangle of* $\triangle A_0B_0C_0$ (see Figure 1). If $\triangle A_1B_1C_1$ is the $(p\!:\!q)$-subtriangle of $\triangle A_0B_0C_0$ and if $\triangle A_2B_2C_2$ is the $(p\!:\!q)$-subtriangle of $\triangle A_1B_1C_1$, then $\triangle A_2B_2C_2$ is called *the* $(p\!:\!q)^2$*-subtriangle of* $\triangle A_0B_0C_0$. This is extended inductively in the obvious way to define the $(p\!:\!q)^k$-subtriangle of $\triangle A_0B_0C_0$ for any natural number k.

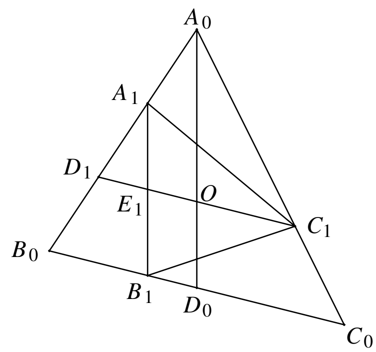


Figure 1.

Theorem 2.2. *The* $(2\!:\!1)$*-chord from* A_0 *in the triangle* $\triangle A_0B_0C_0$ *contains the median from* C_1 *in* $\triangle A_1B_1C_1$*, its* $(1\!:\!2)$*-subtriangle. Dually, the median from* A_0 *in* $\triangle A_0B_0C_0$ *contains the* $(2\!:\!1)$*-chord from* C_1 *in* $\triangle A_1B_1C_1$*. Analogous results hold for the appropriate other chords and medians. Consequently, the centroids of both triangles coincide.*

Proof. Let D_1 be the $(2\!:\!1)$-point from A_0 to B_0. To see that the segment $\overline{C_1D_1}$ bisects the segment $\overline{A_1B_1}$ is another easy exercise in similarity. Thus the first statement follows. Next, if D_0 is the midpoint of the segment $\overline{B_0C_0}$, and O is the intersection of segment $\overline{A_0D_0}$ and segment $\overline{C_1D_1}$, it is the centroid of $\triangle A_0B_0C_0$ by Theorem 2.1. Another application of the first part of this theorem (at the vertex B_0, say) shows that O is also the centroid of $\triangle A_1B_1C_1$. Moreover, if E_1 is the intersection of segment $\overline{A_1B_1}$ and segment $\overline{C_1D_1}$, then $C_1O : OE_1 = 2\!:\!1$ (because O is the centroid of $\triangle A_1B_1C_1$). It follows that the segment $\overline{A_0D_0}$ contains the $(2\!:\!1)$-chord from C_1 in $\triangle A_1B_1C_1$. ■

There is a more general result for part of this theorem—see Theorem 3.1. One of the classical results in geometry is that the $(1\!:\!1)$-subtriangle of $\triangle ABC$ is similar to $\triangle ABC$. In a similar vein (but rather more difficult to prove) we have the following.

Theorem 2.3. *The* $(1\!:\!2)^2$*-subtriangle of* $\triangle A_0B_0C_0$ *is similar to* $\triangle A_0B_0C_0$*.*

Proof. (See Figure 2.) By Theorem 2.2, the $(2\!:\!1)$-chord from B_2 in $\triangle A_2B_2C_2$ is a subsegment of the $(2\!:\!1)$-chord from A_0 in $\triangle A_0B_0C_0$, and the analogous statement holds for the other $(2\!:\!1)$-chords. From this it follows that the line containing A_2 and C_2 is parallel to the line containing B_0 and C_0, etc. Again by appeal to Theorem 2.2, the median $\overline{B_2D_2}$ in $\triangle A_2B_2C_2$ is a subsegment of the median $\overline{A_0D_0}$ of $\triangle A_0B_0C_0$, so that $\triangle A_0B_0D_0 \sim \triangle B_2C_2D_2$ and $\triangle A_0C_0D_0 \sim \triangle B_2A_2D_2$ (The line containing A_0 and D_0 is a transversal between three pairs of parallel lines.) Consequently $\triangle A_0B_0C_0 \sim \triangle B_2C_2A_2$. ■

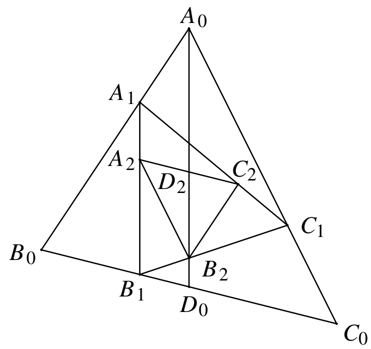


Figure 2.

3. A MATRIX THEORY VIEW. The preceding arguments are purely geometric: Euclid himself would have had no difficulty understanding them. We know that the $(1\!:\!1)^1$-subtriangle and the $(1\!:\!2)^2$-subtriangle of $\triangle ABC$ are both similar to $\triangle ABC$. This naturally suggests:

The Question. *Is the $(1\!:\!n)^n$-subtriangle of $\triangle ABC$ similar to $\triangle ABC$ for each natural number n?*

To answer this we leave Euclid (and Eudoxus) behind and move forward to mathematics that Cayley would have understood.

The Cartesian plane is a model for Euclidean geometry. Thus a theorem in Euclidean geometry will be true in the Cartesian plane, and (contrapositively) a statement that is false in the Cartesian plane will not be true in Euclidean geometry. In the Cartesian plane we can represent $\triangle A_0B_0C_0$ by a 3×2 matrix

$$\mathbf{T}_0 = \begin{pmatrix} x_a & y_a \\ x_b & y_b \\ x_c & y_c \end{pmatrix}, \tag{3.1}$$

where the rows $\mathbf{a}_0 = (x_a, y_a)$, $\mathbf{b}_0 = (x_b, y_b)$, and $\mathbf{c}_0 = (x_c, y_c)$ of $\mathbf{T}_0$ are the respective coordinates of A_0, B_0, and C_0. We can then obtain $\mathbf{T}_1$, the coordinate matrix for the $(p\!:\!q)$-subtriangle of $\triangle A_0B_0C_0$, by matrix multiplication $\mathbf{T}_1 = \mathbf{S}\mathbf{T}_0$, where

$$\mathbf{S} = \frac{1}{p+q}\begin{pmatrix} q & p & 0 \\ 0 & q & p \\ p & 0 & q \end{pmatrix}. \tag{3.2}$$

More generally,

$$\mathbf{T}_k = \mathbf{S}^k\mathbf{T}_0 \tag{3.3}$$

is the $(p\!:\!q)^k$-subtriangle of $\triangle A_0B_0C_0$. Using this we can prove the following more general result for part of Theorem 2.2.

Theorem 3.1. *A triangle $\triangle A_0B_0C_0$ and its $(p\!:\!q)$-subtriangle $\triangle A_1B_1C_1$ have a common centroid for all natural numbers p and q.*

Proof. If $\mathbf{a}_i$, $\mathbf{b}_i$, and $\mathbf{c}_i$ are the respective coordinates of A_i, B_i, and C_i for $i = 0, 1$, then

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \\ \mathbf{c}_1 \end{pmatrix} = \frac{1}{p+q} \begin{pmatrix} q & p & 0 \\ 0 & q & p \\ p & 0 & q \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \\ \mathbf{c}_0 \end{pmatrix} = \frac{1}{p+q} \begin{pmatrix} q\,\mathbf{a}_0 + p\,\mathbf{b}_0 \\ q\,\mathbf{b}_0 + p\,\mathbf{c}_0 \\ q\,\mathbf{c}_0 + p\,\mathbf{a}_0 \end{pmatrix},$$

so

$$\frac{\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{c}_1}{3} = \frac{\mathbf{a}_0 + \mathbf{b}_0 + \mathbf{c}_0}{3}.$$

This establishes the theorem, because the centroid χ of the triangle defined by $\mathbf{T}_0$ is

$$\chi = \frac{\mathbf{a}_0 + \mathbf{b}_0 + \mathbf{c}_0}{3}. \tag{3.4}$$

This is an exercise in many calculus texts, but it is also a consequence of the result in classical geometry stating that the centroid is the point on any median that is two-thirds the distance to the midpoint of the opposite side. As a result, the centroid is given by

$$\frac{\mathbf{a}_0}{3} + \left(\frac{2}{3}\right)\left(\frac{\mathbf{b}_0 + \mathbf{c}_0}{2}\right) = \frac{\mathbf{a}_0 + \mathbf{b}_0 + \mathbf{c}_0}{3}. \qquad \blacksquare$$

Taking $p = 1$ and $q = 2$ gives us the following alternate way to prove Theorem 2.3, which is valid in the Cartesian plane.

Alternate proof of Theorem 2.3. The coordinates for the $(1:2)^2$-subtriangle $\triangle A_2B_2C_2$ in $\triangle A_0B_0C_0$ are obtained as

$$\begin{pmatrix} \mathbf{a}_2 \\ \mathbf{b}_2 \\ \mathbf{c}_2 \end{pmatrix} = \mathbf{S}^2\mathbf{T}_0 = \frac{1}{9} \begin{pmatrix} 4 & 4 & 1 \\ 1 & 4 & 4 \\ 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \\ \mathbf{c}_0 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 4\mathbf{a}_0 + 4\mathbf{b}_0 + \mathbf{c}_0 \\ \mathbf{a}_0 + 4\mathbf{b}_0 + 4\mathbf{c}_0 \\ 4\mathbf{a}_0 + \mathbf{b}_0 + 4\mathbf{c}_0 \end{pmatrix}.$$

Consequently,

$$\|\mathbf{a}_2 - \mathbf{b}_2\| = \frac{\|\mathbf{c}_0 - \mathbf{a}_0\|}{3}, \quad \|\mathbf{b}_2 - \mathbf{c}_2\| = \frac{\|\mathbf{a}_0 - \mathbf{b}_0\|}{3}, \quad \|\mathbf{c}_2 - \mathbf{a}_2\| = \frac{\|\mathbf{b}_0 - \mathbf{c}_0\|}{3},$$

and thus $\triangle A_0B_0C_0 \sim \triangle B_2C_2A_2$. $\blacksquare$

We return to the question of whether the $(1:n)^n$-subtriangle of $\triangle ABC$ is similar to $\triangle ABC$ for each natural number n. Drawings using Geometer's Sketchpad seem to support this idea. Unfortunately, careful analysis using Maple shows that it is not correct—the $(1:n)^n$-subtriangle is generally *not* similar to the original triangle when $n > 2$.

While our experiments uncovered the flaw in the conjecture, they prompted us to question the limiting behavior of $(p:q)$-subtriangles. For example, we can successively build $(p:q)$-subtriangles for which the ratio p/q remains fixed at each iteration, or we can fix p and allow q to vary at each step. Both of these limiting processes produce interesting results. Examination of the structure of the matrix $\mathbf{S}$ in (3.2) reveals that these problems are really questions concerning the limiting behavior of stochastic matrices, and thus limiting properties of $(p:q)$-subtriangles carry us deeper into matrix theory.

4. SUBTRIANGLES AND STOCHASTIC MATRICES. We examine the limiting behavior of $(p:q)$-subtriangles by means of the equation $\mathbf{T}_k = \mathbf{S}^k\mathbf{T}_0$, where $\mathbf{T}_0$ and $\mathbf{S}$ are as defined in (3.1) and (3.2). To this end, it is helpful to recall some facts concerning stochastic matrices.

Stochastic matrices: a quick review. Let $\mathbf{S}$ be a *stochastic* matrix—i.e., $\mathbf{S}$ is a square matrix such that $\mathbf{S} \geq \mathbf{0}$ (entrywise) and each row sum of $\mathbf{S}$ is 1. We will make use the following features of stochastic matrices. See Meyer [**3**, chap. 8] for proofs and more detailed discussions.

- $\mathbf{S}$ is *doubly stochastic* if each column sum (in addition to each row sum) is 1.
- $\mathbf{S}$ is *irreducible* whenever there is no permutation matrix $\mathbf{Q}$ such that

$$\mathbf{Q}^T\mathbf{S}\mathbf{Q} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix},$$

  where $\mathbf{X}$ and $\mathbf{Z}$ are square matrices.
- If $\mathbf{u}$ is a column vector of ones, then $\mathbf{Su} = \mathbf{u}$, so $(1, \mathbf{u})$ is an eigenpair for $\mathbf{S}$.
- All eigenvalues of $\mathbf{S}$ are contained in or on the unit circle in the complex plane.
- $\mathbf{S}$ is *primitive* when $\mathbf{S}$ is irreducible and $\lambda = 1$ is the *only* eigenvalue of $\mathbf{S}$ on the unit circle.
- $\mathbf{S}$ is primitive if and only if $\mathbf{S}^k > \mathbf{0}$ for some positive integer k. (4.1)
- If $\mathbf{S}_{m\times m}$ is primitive, then $\lim_{n\to\infty} \mathbf{S}^n$ exists and is given by

$$\lim_{n\to\infty} \mathbf{S}^n = \frac{\mathbf{x}\mathbf{y}^T}{\mathbf{y}^T\mathbf{x}},$$

  where $\mathbf{x}$ and $\mathbf{y}$ are respective right-hand and left-hand eigenvectors for $\mathbf{S}$ that are associated with the eigenvalue 1. If $\mathbf{S}$ is doubly stochastic, then

$$\lim_{n\to\infty} \mathbf{S}^n = \frac{1}{m}\mathbf{u}\mathbf{u}^T = \frac{1}{m}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{4.2}$$

We first consider the limiting nature of the $(p:q)^k$-subtriangles of $\triangle A_0B_0C_0$ when the ratio p/q is fixed. This case is fairly intuitive, and most of us would guess the correct answer (especially after Theorem 3.1).

Theorem 4.1. *If p/q is fixed, then starting with $\triangle A_0B_0C_0$ the sequence $\{\triangle A_kB_kC_k\}_{k=0}^{\infty}$ of $(p:q)^k$-subtriangles converges to the centroid of $\triangle A_0B_0C_0$.*

Proof. The coordinates of the vertices of $\triangle A_kB_kC_k$ are given by the rows of $\mathbf{T}_k = \mathbf{S}^k\mathbf{T}_0$, where $\mathbf{S}$ is the doubly stochastic matrix given in (3.2) and $\mathbf{T}_0$ is as described in (3.1). Since $\mathbf{S}^2 > 0$, it follows from (4.1) that $\mathbf{S}$ is primitive, and consequently (4.2) guarantees that

$$\lim_{k\to\infty} \mathbf{S}^k = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

As pointed out in (3.4), the centroid χ of the triangle defined by

$$\mathbf{T}_0 = \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \\ \mathbf{c}_0 \end{pmatrix}$$

is given by

$$\chi = \frac{\mathbf{a}_0 + \mathbf{b}_0 + \mathbf{c}_0}{3},$$

so the coordinates of the vertices of the limiting $(p:q)$-subtriangle are obtained from

$$\lim_{k\to\infty} \mathbf{T}_k = \lim_{k\to\infty} \mathbf{S}^k\mathbf{T}_0 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \\ \mathbf{c}_0 \end{pmatrix} = \begin{pmatrix} \chi \\ \chi \\ \chi \end{pmatrix}. \qquad \blacksquare$$

If p/q is not fixed, then the limiting behavior of the sequence $\triangle A_k B_k C_k$ of $(p:q)^k$-subtriangles is more interesting. In particular, the sequence $\triangle A_k B_k C_k$ need not converge to the centroid of $\triangle A_0 B_0 C_0$—in fact, the limit is not necessarily a point. For example, when $p = 1$ and $q = n$, the limit of the $(1:n)^n$-subtriangles as $n \to \infty$ is a triangle. This is illustrated in Figure 3 for $n = 5$, $n = 10$, and $n = 50$.

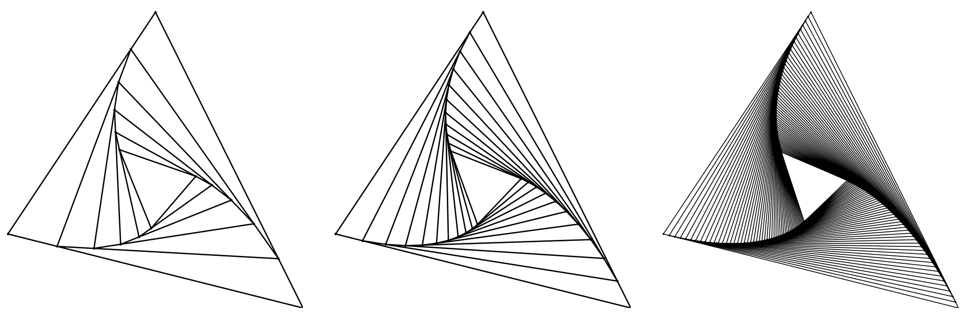


Figure 3.

The limit that is suggested by Figure 3 has a surprisingly elegant representation.

Theorem 4.2. *If* $\mathbf{T}_0$ *is the coordinate matrix of* $\triangle A_0 B_0 C_0$ *as described in (3.1) and if* $\mathbf{T}_n$ *is the coordinate matrix for the* $(1:n)^n$*-subtriangle, then the coordinate matrix for the limiting triangle is* $\lim_{n\to\infty} \mathbf{T}_n = e^{\mathbf{P}-\mathbf{I}}\mathbf{T}_0$, *where*

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. It follows from (3.2) and (3.3) that

$$\lim_{n\to\infty} \mathbf{T}_n = \lim_{n\to\infty} \mathbf{S}^n\mathbf{T}_0,$$

where

$$\mathbf{S} = \frac{1}{n+1}\begin{pmatrix} n & 1 & 0 \\ 0 & n & 1 \\ 1 & 0 & n \end{pmatrix} = \frac{n}{n+1}\left(\mathbf{I} + \frac{\mathbf{P}}{n}\right).$$

Since $n/(n+1) = 1/(1+n^{-1})$ and $\lim_{n\to\infty}(1+n^{-1})^n = \mathrm{e}$, it follows that

$$\left[\frac{n}{n+1}\right]^n \to \mathrm{e}^{-1}.$$

Furthermore, matrix limits act the same as scalar limits insofar as the exponential function is involved, so $\lim_{n\to\infty}(\mathbf{I}+\mathbf{P}/n)^n = \mathrm{e}^{\mathbf{P}}$. Consequently

$$\lim_{n\to\infty} \mathbf{S}^n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \lim_{n\to\infty}\left(\mathbf{I}+\frac{\mathbf{P}}{n}\right)^n = \mathrm{e}^{-1}\mathrm{e}^{\mathbf{P}} = \mathrm{e}^{\mathbf{P}-\mathbf{I}}. \qquad \blacksquare$$

Explicitly producing the coordinates of the limiting subtriangle requires evaluation of the matrix exponential $\mathrm{e}^{\mathbf{P}-\mathbf{I}} = \mathrm{e}^{-1}\mathrm{e}^{\mathbf{P}}$. The standard approach is to diagonalize $\mathbf{P}$ with a similarity transformation $\mathbf{F}^{-1}\mathbf{P}\mathbf{F} = \mathbf{D}$, which in our case is the *Fourier matrix* of order three

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \lambda & \lambda^2 \\ 1 & \lambda^2 & \lambda^4 \end{pmatrix},$$

where

$$\lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, \qquad \mathbf{F}^{-1} = (1/3)\overline{\mathbf{F}}, \qquad \mathbf{D} = \begin{pmatrix} 1 & & \\ & \lambda & \\ & & \overline{\lambda} \end{pmatrix} \tag{4.3}$$

(see Meyer [**3**, pp. 357, 379]). However, a more elementary approach to computing $\mathbf{e}^{\mathbf{P}-\mathbf{I}}$ is presented in section 7. It is more straightforward than diagonalization and better suits our purposes (explicit formulas are given section 7).

Returning to triangles, consider now the limiting behavior of the $(p\!:\!q)^q$-subtriangles when p is fixed and $q \to \infty$. For example, fixing $p = 2$, we depict the situations for $q = 5$, $q = 10$, and $q = 50$ in Figure 4.

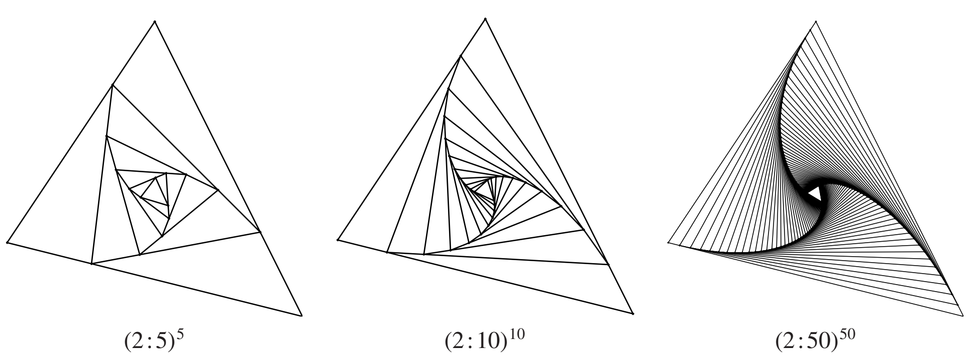


Figure 4.

It is not clear from these drawings if the limit is a point or a triangle, but the same technique used to prove Theorem 4.2 generalizes to produce the answer.

Theorem 4.3. *Let $\mathbf{T}_0$ be the coordinate matrix of $\triangle A_0B_0C_0$ as described in (3.1), and let p be a fixed natural number. If $\mathbf{T}_q$ is the coordinate matrix for the $(p\!:\!q)^q$-sub-*

triangle, then the coordinate matrix for the limiting triangle is $\lim_{q\to\infty} \mathbf{T}_q = \mathrm{e}^{p(\mathbf{P}-\mathbf{I})}\mathbf{T}_0$, *where*

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. Just as in the case of Theorem 4.2, write $\lim_{q\to\infty} \mathbf{T}_q = \lim_{q\to\infty} \mathbf{S}^q \mathbf{T}_0$, but now use

$$\mathbf{S} = \frac{q}{p+q}\left(\mathbf{I} + \frac{p}{q}\mathbf{P}\right),$$

and note that

$$\lim_{q\to\infty}\left(\frac{q}{p+q}\right)^q = \mathrm{e}^{-p}, \qquad \lim_{q\to\infty}\left(\mathbf{I} + \frac{p}{q}\mathbf{P}\right)^p = \mathrm{e}^{p\mathbf{P}}. \qquad \blacksquare$$

As mentioned earlier, the $(1\!:\!n)^n$-subtriangle is generally not similar to the original triangle $\triangle A_0B_0C_0$ when $n > 2$. Expressions given in section 7 can be used to verify that the same is true for the limiting case as well, and analogous statements hold for the subtriangles in Theorem 4.3. However, if a $(p\!:\!q)$-subtriangle is followed by a $(q:p)$-subtriangle, then an elegant similarity result is possible.

Theorem 4.4. *The* $(q:p)(p\!:\!q)$*-subtriangle (obtained by constructing the* $(q:p)$*-subtriangle of the* $(p\!:\!q)$*-subtriangle of* $\triangle A_0B_0C_0$*) is similar to* $\triangle A_0B_0C_0$*.*

Proof. Let $\mathbf{T}_0$ be the coordinate matrix for $\triangle A_0B_0C_0$. If $\mathbf{S}$ denotes the product

$$\begin{aligned}\mathbf{S} &= \frac{1}{(p+q)^2}\begin{pmatrix} p & q & 0 \\ 0 & p & q \\ q & 0 & p \end{pmatrix}\begin{pmatrix} q & p & 0 \\ 0 & q & p \\ p & 0 & q \end{pmatrix} \\ &= \frac{1}{(p+q)^2}\begin{pmatrix} pq & p^2+q^2 & pq \\ pq & pq & p^2+q^2 \\ p^2+q^2 & pq & pq \end{pmatrix},\end{aligned} \tag{4.4}$$

then the coordinate matrix $\mathbf{T}_2$ for the $(q:p)(p\!:\!q)$-subtriangle is

$$\begin{pmatrix} \mathbf{a}_2 \\ \mathbf{b}_2 \\ \mathbf{c}_2 \end{pmatrix} = \mathbf{T}_2 = \mathbf{S}\mathbf{T}_0 = \frac{1}{(p+q)^2}\begin{pmatrix} pq & p^2+q^2 & pq \\ pq & pq & p^2+q^2 \\ p^2+q^2 & pq & pq \end{pmatrix}\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \\ \mathbf{c}_0 \end{pmatrix}.$$

For $\xi = (p^2 - pq + q^2)/(p+q)^2$, it is apparent that

$$\|\mathbf{a}_2 - \mathbf{b}_2\| = \xi\|\mathbf{b}_0 - \mathbf{c}_0\|, \quad \|\mathbf{b}_2 - \mathbf{c}_2\| = \xi\|\mathbf{a}_0 - \mathbf{c}_0\|, \quad \|\mathbf{c}_2 - \mathbf{a}_2\| = \xi\|\mathbf{a}_0 - \mathbf{b}_0\|,$$

so $\triangle A_0B_0C_0 \sim \triangle C_2A_2B_2$. $\blacksquare$

The iterated sequence of $(q:p)(p\!:\!q)$-subtriangles has a simple limit.

Theorem 4.5. *The sequence obtained by iteratively constructing* $(q:p)(p:q)$*-subtriangles converges to the centroid of the original triangle* $\triangle A_0B_0C_0$*.*

Proof. Let $\mathbf{S}$ be the matrix given in (4.4), and notice that $\mathbf{S}$ is a doubly stochastic matrix. Application of (4.2) yields

$$\lim_{n\to\infty} \mathbf{S}^n = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

which means that

$$\lim_{n\to\infty} \mathbf{T}_n = \lim_{n\to\infty} \mathbf{S}^n \mathbf{T}_0 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \\ \mathbf{c}_0 \end{pmatrix} = \begin{pmatrix} \chi \\ \chi \\ \chi \end{pmatrix},$$

where $\chi = (\mathbf{a}_0 + \mathbf{b}_0 + \mathbf{c}_0)/3$. ∎

5. GENERALIZATIONS. An interesting generalization occurs by considering a sequence of subtriangles where at the nth stage we use a ratio of $(1 : f(n))$, where f is a (natural number valued) function of n. Beginning with a triangle whose coordinate matrix is $\mathbf{T}_0$, successively generate subtriangles with coordinate matrices $\mathbf{T}_n$ in which $\mathbf{T}_n = \mathbf{S}_n \mathbf{T}_{n-1}$, where

$$\mathbf{S}_n = \frac{1}{f(n)+1}\begin{pmatrix} f(n) & 1 & 0 \\ 0 & f(n) & 1 \\ 1 & 0 & f(n) \end{pmatrix} = \frac{1}{f(n)+1}\big(f(n)\mathbf{I} + \mathbf{P}\big).$$

If a limiting triangle exists, what is its coordinate matrix? The solution is conceptually straightforward because each $\mathbf{S}_k$ is diagonalized by the same matrix, namely, the Fourier matrix of order 3 given in (4.3). Specifically,

$$\mathbf{F}^{-1}\mathbf{S}_k\mathbf{F} = \mathbf{D}_k = \frac{1}{f(k)+1}\big(f(k)\mathbf{I} + \mathbf{D}\big),$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & & \\ & \lambda & \\ & & \overline{\lambda} \end{pmatrix}, \qquad \lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}.$$

Consequently,

$$\mathbf{T}_n = \prod_{k=1}^{n} \mathbf{S}_k \mathbf{T}_0 = \mathbf{F}\left(\prod_{k=1}^{n} \mathbf{D}_k\right)\mathbf{F}^{-1}\mathbf{T}_0 = \mathbf{F}\begin{pmatrix} 1 & & \\ & \prod_{k=1}^{n} \frac{f(k)+\lambda}{f(k)+1} & \\ & & \prod_{k=1}^{n} \frac{f(k)+\overline{\lambda}}{f(k)+1} \end{pmatrix}\mathbf{F}^{-1}\mathbf{T}_0,$$

so the question boils down to analyzing the limits

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{f(k)+\lambda}{f(k)+1}, \qquad \lim_{n\to\infty} \prod_{k=1}^{n} \frac{f(k)+\overline{\lambda}}{f(k)+1}. \tag{5.1}$$

Existence of these limits is resolved by writing

$$\frac{f(k)+\lambda}{f(k)+1} = 1 + a_k,$$

where $a_k = (\lambda - 1)/(f(k)+1)$, and by applying the result from classical analysis that states that $\prod_{k=1}^{\infty}(1 + a_k)$ is absolutely convergent if and only if $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (see Whittaker and Watson [**4**, p. 32]). It is remarkable that not only can the existence of the limit be guaranteed for some cases, but in fact the limit can be evaluated.

Example 5.1. If $f(n) = n$ for $n > 1$, then the limiting $(1:n)$-subtriangle of $\triangle A_0B_0C_0$ is the centroid.

Proof. The limit in (5.1) becomes

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{k+\lambda}{k+1} = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{k}{k+1}\left(1+\frac{\lambda}{k}\right).$$

It can be seen that this limit evaluates to zero by observing that

$$\left|1+\frac{\lambda}{k}\right|^2 = \left(1+\frac{\lambda}{k}\right)\left(1+\frac{\overline{\lambda}}{k}\right) = 1-\frac{1}{k}+\frac{1}{k^2} < 1,$$

from which we infer that

$$\left|\prod_{k=1}^{n}\frac{k+\lambda}{k+1}\right| < \prod_{k=1}^{n}\left(\frac{k}{k+1}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1} = \frac{1}{n+1}\to 0.$$

Accordingly,

$$\lim_{n\to\infty}\mathbf{T}_n = \mathbf{F}\begin{pmatrix}1 & & \\ & 0 & \\ & & 0\end{pmatrix}\mathbf{F}^{-1}\mathbf{T}_0 = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_0\\ \mathbf{b}_0\\ \mathbf{c}_0\end{pmatrix} = \begin{pmatrix}\chi\\ \chi\\ \chi\end{pmatrix}. \qquad \blacksquare$$

Example 5.2. If $f(n) = n^x$, where $x > 1$ is an integer, then the limiting $(1:n^x)$-subtriangle of $\triangle A_0B_0C_0$ is a triangle whose coordinates can be computed as follows. Results concerning the gamma function from Whittaker and Watson [**4**, pp. 238–239] ensure that, if $\{\lambda_1, \lambda_2, \dots, \lambda_x\}$ are the xth roots of $-\lambda = 1/2 - (\sqrt{3}/2)\mathrm{i}$ and $\{\nu_1, \nu_2, \dots, \nu_x\}$ are the xth roots of -1, then

$$\prod_{n=1}^{\infty}\frac{n^x+\lambda}{n^x+1} = \prod_{i=1}^{x}\frac{\Gamma(1-\nu_i)}{\Gamma(1-\lambda_i)}.$$

Explicit evaluation is possible for the case $x = 2$ by making use of the identity

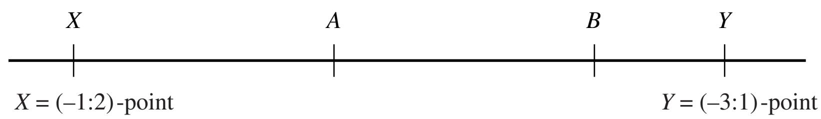
$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$$

to write

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{n^2+\lambda}{n^2+1} &= \frac{\Gamma(1-\mathrm{i})\Gamma(1+\mathrm{i})}{\Gamma(1-(\sqrt{3}+\mathrm{i})/2)\Gamma(1+(\sqrt{3}+\mathrm{i})/2)}\\ &= \frac{\sinh(\lambda\pi)}{\lambda\sinh(\pi)} \approx -.01406 - .20190\mathrm{i}.\end{aligned}$$

6. SUPERTRIANGLES. Instead of restricting ourselves to internal points of division of the sides of a triangle we can consider external divisions of the (extended) sides of a triangle.

Definition 6.1. Let p and q be positive integers. If $p < q$, the $(-p:q)$-point (= the $(p:-q)$-point) X on the line $\overleftrightarrow{AB}$ is the point such that A is between X and B and $XA:XB = p:q$ (note that $p < q$, since the distance XA is always less than XB). If $p > q$, the $(-p:q)$-point (= the $(p:-q)$-point) on the line $\overleftrightarrow{AB}$ is the point Y such that B is between A and Y and $AY:BY = p:q$ (note that $p > q$, since the distance AY is always greater than BY). The $(-p:q)$-point of $\overleftrightarrow{AB}$ is the same as the $(-q:p)$-point of $\overleftrightarrow{BA}$.



Definition 6.2. In triangle $\triangle A_0B_0C_0$, let A_1, B_1, and C_1 be the $(-p:q)$-points of $\overleftrightarrow{A_0B_0}$, $\overleftrightarrow{B_0C_0}$, and $\overleftrightarrow{C_0A_0}$, respectively. Triangle $\triangle A_1B_1C_1$ is the $(-p:q)$-*supertriangle* of $\triangle A_0B_0C_0$, and the $(-p:q)$-supertriangle of $\triangle A_1B_1C_1$ is called the $(-p:q)^2$-*supertriangle* of $\triangle A_0B_0C_0$. Iterating this concept defines the $(-p:q)^k$-supertriangle.

For example, the $(-1:3)$-supertriangle is shown in Figure 5.

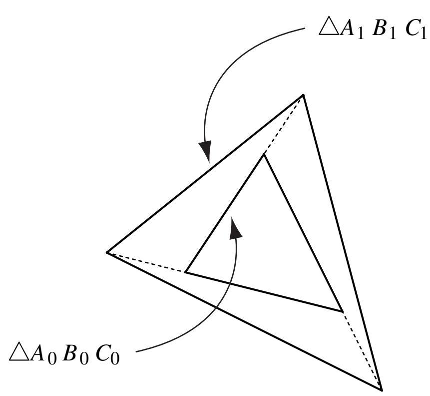


Figure 5. $(-1:3)$-Supertriangle.

If $\mathbf{T}_0$ is the coordinate matrix of $\triangle A_0B_0C_0$ as presented in (3.1), the coordinate matrix of the $(-p:q)$-supertriangle is given by $\mathbf{T}_1 = \mathbf{RT}_0$, where

$$\mathbf{R} = \frac{1}{q-p}\begin{pmatrix} q & -p & 0 \\ 0 & q & -p \\ -p & 0 & q \end{pmatrix}. \tag{6.1}$$

It is easy to prove the following analogue Theorem 3.1.

Theorem 6.1. *A triangle $\triangle A_0B_0C_0$ and its $(-p:q)$-supertriangle $\triangle A_1B_1C_1$ share the same centroid.*

Following the same procedure as in section 4, we consider what happens to the sequence of $(-p:q)^k$-supertriangles for fixed p/q as $k \to \infty$.

Theorem 6.2. *Start with an initial triangle $\triangle A_0B_0C_0$, and let p/q be fixed. The sequence of $(-p:q)^k$-supertriangles grows without limit.*

Proof. We need to investigate what happens to $\mathbf{R}^k$ as $k \to \infty$, where $\mathbf{R}$ is the matrix in (6.1). The eigenvalues of $\mathbf{R}$ are 1 and

$$\frac{-q + (-1 \pm \sqrt{3}\mathrm{i})(p/2)}{q - p}.$$

Because $q - p \geq 1$, it is easy to see that the complex eigenvalues have absolute value greater than 1, which implies that $\mathbf{R}^k$ becomes unbounded as $k \to \infty$. ∎

We now turn to analyzing what happens to the $(-1:n)^n$-supertriangles as $n \to \infty$. As seen in Figure 6, experiments suggest that there will be a limiting supertriangle.

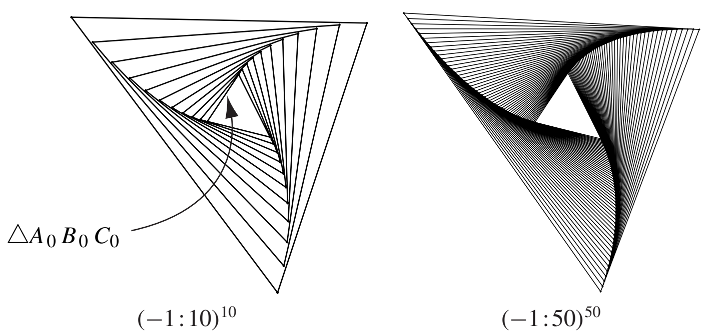


Figure 6.

Recall from Theorem 4.2 that, if $\mathbf{T}_0$ is the coordinate matrix of an initial triangle, then the coordinate matrix for the limiting $(1:n)^n$-subtriangle is $\lim_{n\to\infty} \mathbf{T}_n = \mathrm{e}^{\mathbf{P}-\mathbf{I}}\mathbf{T}_0$. It is interesting (and rather surprising) that the limiting $(-1:n)^n$-supertriangle is essentially given by the inverse of the limiting $(1:n)^n$-subtriangle. To be specific, the following is true.

Theorem 6.3. *If $\mathbf{T}_0$ is the coordinate matrix of $\triangle A_0B_0C_0$ and if $\mathbf{T}_n$ is the coordinate matrix for the $(-1:n)^n$-supertriangle, then the coordinate matrix for the limiting triangle is $\lim_{n\to\infty} \mathbf{T}_n = e^{\mathbf{I}-\mathbf{P}}\,\mathbf{T}_0$, where*

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. We have $\lim_{n\to\infty} \mathbf{T}_n = \lim_{n\to\infty} \mathbf{R}^n\mathbf{T}_0$, where

$$\mathbf{R} = \frac{1}{n-1}\begin{pmatrix} n & -1 & 0 \\ 0 & n & -1 \\ -1 & 0 & n \end{pmatrix}.$$

Moreover,

$$\lim_{n\to\infty} \mathbf{R}^n = \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n \lim_{n\to\infty} \left(\mathbf{I} - \frac{\mathbf{P}}{n}\right)^n = ee^{-\mathbf{P}} = e^{\mathbf{I}-\mathbf{P}}. \qquad \blacksquare$$

Example 6.1. As $n \to \infty$, the limiting triangle of the $(-1\!:\!n^x)^n$-supertriangles for any positive integer $x > 1$ can be obtained in a manner similar to that used in (5.1) and Example 5.2. In particular, if

$$\mathbf{S}_k = \frac{1}{k^x - 1}(k^x\mathbf{I} - \mathbf{P}) \qquad (k > 1),$$

then the coordinate matrix for the $(-1\!:\!n^x)^n$-supertriangle is

$$\mathbf{T}_n = \prod_{k=2}^{n} \mathbf{S}_k \mathbf{T}_0 = \mathbf{F}\left(\prod_{k=2}^{n} \mathbf{D}_k\right)\mathbf{F}^{-1}\mathbf{T}_0 = \mathbf{F}\begin{pmatrix} 1 & & \\ & \prod_{k=2}^{n} \frac{k^x-\lambda}{k^x-1} & \\ & & \prod_{k=2}^{n} \frac{k^x-\overline{\lambda}}{k^x-1} \end{pmatrix}\mathbf{F}^{-1}\mathbf{T}_0,$$

so the question boils down to evaluating the products

$$\Pi_x = \prod_{k=2}^{\infty} \frac{k^x - \lambda}{k^x - 1}, \qquad \overline{\Pi_x} = \prod_{k=2}^{\infty} \frac{k^x - \overline{\lambda}}{k^x - 1}, \tag{6.2}$$

where $\lambda = -1/2 + \mathrm{i}\sqrt{3}/2$. For $j = 1, 2, \ldots, x$ let λ_j denote the xth roots of λ, and let α_j be the xth roots of unity. If $a \neq 1$ is a positive real number, then the xth roots of a are given by $\alpha'_j = \sqrt[x]{a}\,\alpha_j$. If we set

$$\Pi_x(a) = \prod_{k=1}^{\infty} \frac{k^x - \lambda}{k^x - a} = \frac{\Gamma(1-\alpha'_1)\Gamma(1-\alpha'_2)\cdots\Gamma(1-\alpha'_x)}{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)\cdots\Gamma(1-\lambda_x)},$$

$$G = \frac{\Gamma(1-\alpha'_2)\cdots\Gamma(1-\alpha'_x)}{\Gamma(1-\lambda_1)\cdots\Gamma(1-\lambda_x)},$$

then continuity of the gamma function along with $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin(\pi z)$ yields

$$\begin{aligned}
\Pi_x &= \lim_{a\to 1} \frac{1-a}{1-\lambda}\Pi_x(a) = \lim_{a\to 1}\left(\frac{1-a}{1-\lambda}\right)\left(\frac{1+\alpha'_1}{1+\alpha'_1}\right)\Pi_x(a) \\
&= \lim_{a\to 1}\left(\frac{1-a}{1-\lambda}\right)\left(\frac{\pi\sqrt[x]{a}}{\sin(\pi\sqrt[x]{a})}\right)\left(\frac{1}{\Gamma(1+\sqrt[x]{a})}\right) \\
&= \lim_{a\to 1}\left(\frac{\pi\sqrt[x]{a}}{1-\lambda}\right)\left(\frac{1-a}{\sin(\pi\sqrt[x]{a})}\right)\left(\frac{1}{\Gamma(1+\sqrt[x]{a})}\right) G \\
&= \left(\frac{\pi}{1-\lambda}\right)\left(\frac{-1}{x^{-1}\pi(-1)}\right) G = \left(\frac{x}{1-\lambda}\right) G.
\end{aligned}$$

Here L'Hôpital's rule is used to evaluate $\lim_{a\to 1}(1-a)/\sin(\pi\sqrt[x]{a})$. For example, when $x = 2$ we have

$$\Pi_2 = \frac{(3-\sqrt{3}\mathrm{i})\cosh(\pi\sqrt{3}/2)}{3\pi},$$

and the limiting $(-1\!:\!n^2)^n$-supertriangle $\triangle A_\infty B_\infty C_\infty$ is shown in Figure 7.

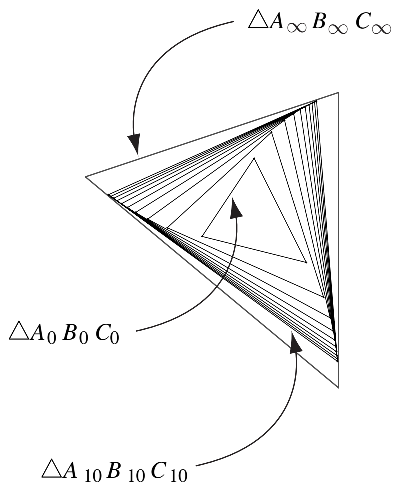


Figure 7. The limiting $(-1:n^2)^n$-supertriangle.

7. EVALUATION OF EXPONENTIALS AND SERIES. In Theorems 4.2, 4.3, and 6.3, the limiting triangles were determined by a matrix exponential of the form $\mathrm{e}^{p(\mathbf{P}-\mathbf{I})}$, where

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and p is a scalar. As shown in what follows, these exponentials are easily evaluated, and thus explicit formulas for the coordinate matrices for the limiting triangles in Theorems 4.2, 4.3, and 6.3 are produced. To compute $\mathrm{e}^{p\mathbf{P}}$ simply note that the periodicity of $\mathbf{P}$ ensures that

$$\mathrm{e}^{p\mathbf{P}} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{P} + \alpha_2 \mathbf{P}^2,$$

where

$$\alpha_0 = \sum_{j=0}^{\infty} \frac{p^{3j}}{(3j)!}, \quad \alpha_1 = \sum_{j=0}^{\infty} \frac{p^{3j+1}}{(3j+1)!}, \quad \alpha_2 = \sum_{j=0}^{\infty} \frac{p^{3j+2}}{(3j+2)!}. \tag{7.1}$$

For the cube root of unity $\lambda = -1/2 + i\sqrt{3}/2$, use $\mathrm{e}^p = \alpha_0 + \alpha_1 + \alpha_2$ together with the real and imaginary parts of $\mathrm{e}^{p\lambda} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$ to produce the 3×3 linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^p \\ \mathrm{e}^{-p/2}\cos(p\sqrt{3}/2) \\ \mathrm{e}^{-p/2}\sin(p\sqrt{3}/2) \end{pmatrix},$$

which is easily solved by inverting the coefficient matrix. The solution is

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \mathrm{e}^p + 2\mathrm{e}^{-p/2}\cos(p\sqrt{3}/2) \\ \mathrm{e}^p - \mathrm{e}^{-p/2}\cos(p\sqrt{3}/2) + \mathrm{e}^{-p/2}\sqrt{3}\sin(p\sqrt{3}/2) \\ \mathrm{e}^p - \mathrm{e}^{-p/2}\cos(\sqrt{3}/2) - \mathrm{e}^{-p/2}\sqrt{3}\sin(\sqrt{3}/2) \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} e^{p} + 2e^{-p/2}\cos(p\sqrt{3}/2) \\ e^{p} + 2e^{-p/2}\sin(p\sqrt{3}/2 - \pi/6) \\ e^{p} - 2e^{-p/2}\sin(p\sqrt{3}/2 + \pi/6) \end{pmatrix},$$

so that

$$e^{p(\mathbf{P}-\mathbf{I})} = e^{-p}\left(\alpha_0\mathbf{I} + \alpha_1\mathbf{P} + \alpha_2\mathbf{P}^2\right) = \frac{1}{e^{p}}\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
$$+ \frac{2e^{-3p/2}}{3}\begin{pmatrix} \cos(p\sqrt{3}/2) & \sin(p\sqrt{3}/2 - \pi/6) & -\sin(p\sqrt{3}/2 + \pi/6) \\ -\sin(p\sqrt{3}/2 + \pi/6) & \cos(p\sqrt{3}/2) & \sin(p\sqrt{3}/2 - \pi/6) \\ \sin(p\sqrt{3}/2 - \pi/6) & -\sin(p\sqrt{3}/2 + \pi/6) & \cos(p\sqrt{3}/2) \end{pmatrix}. \tag{7.2}$$

Consequently, we can make the following observations.

Theorem 7.1.

- *Setting $p = 1$ in (7.2) yields the coordinate matrix of the limiting $(-1:n)^n$-supertriangle as described in Theorem 4.2,*
- *Setting $p = -1$ in (7.2) produces the coordinate matrix of the limiting $(-1:n)^n$-supertriangle as described in Theorem 6.3.*
- *For each fixed natural number p, formula (7.2) yields the coordinate matrix of the limiting $(p:q)^q$-subtriangle as described in Theorem 4.3.*

As a by-product of the calculation leading to (7.2), explicit formulas for the infinite series in (7.1) are produced:

$$\sum_{j=0}^{\infty}\frac{p^{3j}}{(3j)!} = \frac{e^{p}}{3} + \frac{2\cos(p\sqrt{3}/2)}{3\sqrt{e^{p}}},$$
$$\sum_{j=0}^{\infty}\frac{p^{(3j+1)}}{(3j+1)!} = \frac{e^{p}}{3} + \frac{2\sin(p\sqrt{3}/2 - \pi/6)}{3\sqrt{e^{p}}},$$
$$\sum_{j=0}^{\infty}\frac{p^{(3j+2)}}{(3j+2)!} = \frac{e^{p}}{3} - \frac{2\sin(p\sqrt{3}/2 + \pi/6)}{3\sqrt{e^{p}}}.$$

These can also be found in [**2**, #803, #804, p. 150].

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