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Neutron Transport in Two Dissimilar Media with Anisotropic Scattering

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The elementary solutions of the one-speed neutron-transport equation with linearly anisotropic scattering are used in conjunction with Chandrasekhar's invariance principles to solve in a concise manner the Milne problem for two adjoining half-spaces and the critical reactor problem for a reflected slab.

I. INTRODUCTION

The elementary solutions of Case[1] were used by Kuszell[2] to study neutron diffusion for problems defined by the presence of two dissimilar media. That work, however, was limited to isotropic scattering and was left in a somewhat cumbersome final form. Later work by Mendelson and Summerfield[3] added to the general area of multiregion problems; however, it is to the basic work of McCormick[4] and McCormick and Doyas[5] that we must look for the most significant contribution to two-media problems with the effects of anisotropic scattering included. Today in the field of neutron-transport theory many researchers consider the fundamental paper by Case[1] to be the cornerstone of the theory of "exact" solutions. Later Pahor and Zweifel[6] in an elegant paper demonstrated how the work of Chandrasekhar[7] and Case[1] could be coupled and utilized at the same time to obtain in a profitable and concise manner certain results for a variety of single-medium problems.

In a recent Note, Siewert and Burkart[8] demonstrated how the principles of invariance, as developed by Chandrasekhar,[7] could be used effectively to analyze the critical reactor problem for a reflected slab with isotropic scattering. In this paper, we wish to show explicitly the complications that arise when the same critical problem and the Milne problem for two adjoining half-spaces are solved for the case of linearly anisotropic scattering.

II. THE MILNE PROBLEM FOR TWO HALF-SPACES

We consider the one-speed neutron-transport equation for region 1, $x > 0$, and region 2, $x < 0$, written in the familar manner

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[1] K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).

[2] A. KUSZELL, *Acta Phys. Pol.*, **20**, 567 (1961).

[3] M. R. MENDELSON and G. C. SUMMERFIELD, *J. Math. Phys.*, **5**, 668 (1964).

[4] N. J. McCORMICK, *Nucl. Sci. Eng.*, **37**, 243 (1969).

[5] N. J. McCORMICK and R. J. DOYAS, *Nucl. Sci. Eng.*, **37**, 252 (1969).

[6] S. PAHOR and P. F. ZWEIFEL, *J. Math. Phys.*, **10**, 581 (1969).

[7] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London (1950).

[8] C. E. SIEWERT and A. R. BURKART, *Nucl. Sci. Eng.*, **58**, 253 (1975).

$$\mu \frac{\partial}{\partial x} \Psi_\alpha(x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')(1 + b_\alpha \mu\mu')\, d\mu' \quad . \tag{1}$$

Here $\Psi_\alpha(x,\mu)$ denotes the neutron angular density in region α, as a function of position (in optical units) x and the direction cosine of the propagating neutrons, μ. In addition, c_α denotes the mean number of secondary neutrons per collision in region α, and b_α is the coefficient of anisotropic scattering.

For the considered Milne problem, we seek a diverging (as $x \to \infty$) solution of Eq. (1) such that

$$\lim_{x\to\infty} \Psi_1(x,\mu)\exp(-x/\nu_0) < \infty \quad , \tag{2a}$$

$$\lim_{x\to-\infty} \Psi_2(x,\mu) = 0 \tag{2b}$$

and

$$\Psi_1(0,\mu) = \Psi_2(0,\mu) \quad , \quad \mu \in (-1,1) \quad . \tag{2c}$$

Here ν_0 denotes the discrete eigenvalue in region 1. (We use, with only slight modification, the notation of Case and Zweifel,[9] so that many of the basic quantities need not be redefined here.)

Relying on the basic work of McCormick and Kuščer,[10] we can immediately write solutions to Eq. (1) that satisfy the boundary conditions listed as Eqs. (2a) and (2b):

$$\begin{aligned}\Psi_1(x,\mu) = {} & A(\nu_0)\,\phi_1(\nu_0,\mu)\exp(-x/\nu_0) \\ & + \phi_1(-\nu_0,\mu)\exp(x/\nu_0) \\ & + \int_0^1 A(\nu)\,\phi_1(\nu,\mu)\exp(-x/\nu)\,d\nu \end{aligned} \tag{3a}$$

and

$$\begin{aligned}\Psi_2(x,\mu) = {} & B(-\eta_0)\,\phi_2(-\eta_0,\mu)\exp(x/\eta_0) \\ & + \int_0^1 B(-\eta)\,\phi_2(-\eta,\mu)\exp(x/\eta)\,d\eta \quad . \end{aligned} \tag{3b}$$

Here

$$\phi_\alpha(\xi_\alpha,\mu) = \frac{c_\alpha \xi_\alpha}{2} R_\alpha(\xi_\alpha,\mu)\frac{1}{\xi_\alpha - \mu} \quad , \tag{4}$$

where

$$\begin{aligned} R_\alpha(x,y) &= 1 + l_\alpha xy \\ l_\alpha &= b_\alpha(1 - c_\alpha) \\ \xi_1 &= \nu_0 \\ \xi_2 &= \eta_0 \quad , \end{aligned}$$

and where $\xi_\alpha \notin (-1,1)$ is the positive zero of

$$\Lambda_\alpha(z) = 1 - c_\alpha z R_\alpha(z,z)\tanh^{-1}\frac{1}{z} + c_\alpha l_\alpha z^2 \quad . \tag{5}$$

Also,

$$\phi_\alpha(\xi,\mu) = \frac{c_\alpha \xi}{2} R_\alpha(\xi,\mu)\frac{P}{\xi - \mu} + \lambda_\alpha(\xi)\,\delta(\xi - \mu) \quad , \quad \xi \in (-1,1) \quad , \tag{6}$$

with

$$\lambda_\alpha(\xi) = 1 - c_\alpha \xi R_\alpha(\xi,\xi)\tanh^{-1}\xi + c_\alpha l_\alpha \xi^2 \quad . \tag{7}$$

Since the solutions given by Eqs. (3) inherently satisfy Eqs. (2a) and (2b), we need simply to constrain them to obey Eq. (2c), which we choose to write as

$$\Psi_1(0,\mu) = \Psi_2(0,\mu) \quad \text{and} \quad \Psi_1(0,-\mu) = \Psi_2(0,-\mu) \quad , \quad \mu \in (0,1) \quad . \tag{8}$$

At this point we can use the S function of Chandrasekhar[7] to write

$$\Psi_2(0,\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\,\Psi_2(0,-\mu')\,d\mu' \quad , \quad \mu \in (0,1) \quad , \tag{9}$$

where

$$S_2(\mu',\mu) = \frac{c_2 \mu\mu'}{\mu + \mu'}\left[1 - \hat{c}_2(\mu + \mu') - l_2 \mu\mu'\right] \times H_2(\mu')\,H_2(\mu) \quad . \tag{10}$$

Here $H_2(\mu)$ is Chandrasekhar's H function for region 2 and, in general,

$$\hat{c}_\alpha = \frac{c_\alpha l_\alpha \alpha_{\alpha,1}}{2 - c_\alpha \alpha_{\alpha,0}} \quad ,$$

$$\hat{q}_\alpha = \frac{2(1 - c_\alpha)}{2 - c_\alpha \alpha_{\alpha,0}} \quad ,$$

and

$$\alpha_{\alpha,\beta} = \int_0^1 H_\alpha(\mu)\,\mu^\beta\,d\mu \quad . \tag{11}$$

If we now enter Eq. (8) into Eq. (9), we can obtain

$$\Psi_1(0,\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\,\Psi_1(0,-\mu')\,d\mu' \quad , \quad \mu \in (0,1) \quad . \tag{12}$$

We consider that Eq. (12) is the basic equation now to be satisfied, since if $A(\nu_0)$ and $A(\nu)$ are established by Eq. (12), then $B(-\eta_0)$ and $B(-\eta)$ can be obtained immediately from Eq. (8) by using the half-range orthogonality relations of McCormick and Kuščer.[10]

On substituting Eq. (3a) into Eq. (12), we find that we can evaluate the integral over μ' to obtain

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[9] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Company, Reading, Massachusetts (1967).

[10] N. J. McCORMICK and I. KUŠČER, *J. Math. Phys.*, **6**, 1939 (1965).

$$\frac{A(\nu_0)}{H_2(\nu_0)}\left[\phi_1(\nu_0,\mu) - W(\nu_0)\right] + \int_0^1 \frac{A(\nu)}{H_2(\nu)}\left[\phi_1(\nu,\mu) - W(\nu)\right]d\nu = -\frac{1}{H_2(-\nu_0)}\left[\phi_1(-\nu_0,\mu) - W(-\nu_0)\right] \quad , \quad \mu \in (0,1) \quad , \tag{13}$$

where

$$W(\xi) = \frac{1}{2}\frac{c_1\xi}{R_2(\xi,\xi)}\left\{\xi(l_1 - l_2)\left[\hat{q}_2H_2(\xi) - 1\right] - \hat{c}_2R_1(\xi,\xi)\right\} \quad . \tag{14}$$

Equation (13) clearly is a singular integral equation that can be regularized by using the half-range orthogonality relations for one medium (McCormick and Kuščer[10]). Thus, if we multiply Eq. (13) by

$$\left[\phi_1(\nu_0,\mu) + \frac{c_1\nu_0}{2}\hat{c}_1\right]\mu H_1(\mu)$$

and integrate over μ, we find

$$\frac{A(\nu_0)}{H_2(\nu_0)}\left[N_1(\nu_0)H_1(\nu_0) - \nu_0\hat{q}_1W(\nu_0)\right] - \nu_0\hat{q}_1\overline{A} = -\frac{1}{H_2(-\nu_0)}\left[J(-\nu_0,\nu_0) - \nu_0\hat{q}_1W(-\nu_0)\right] \quad , \tag{15}$$

where

$$\overline{A} = \int_0^1 \frac{A(\nu)}{H_2(\nu)}W(\nu)\,d\nu \quad , \tag{16}$$

$$N_1(\nu_0) = \frac{c_1\nu_0^2}{2}R_1(\nu_0,\nu_0)\left[\frac{c_1R_1(\nu_0,\nu_0)}{\nu_0(\nu_0^2 - 1)} - \frac{(1 - c_1)R_1(3\nu_0,\nu_0)}{\nu_0R_1(\nu_0,\nu_0)}\right] \quad , \tag{17}$$

and

$$J(-\nu_0,\xi) = \frac{c_1\nu_0\xi}{2(\nu_0 + \xi)H_1(\nu_0)} \times \left[1 - l_1\nu_0\xi + \hat{c}_1(\nu_0 + \xi)\right] \quad . \tag{18}$$

In a similar manner, we can multiply Eq. (13) by

$$\left[\phi_1(\nu',\mu) + \frac{c_1\nu'}{2}\hat{c}_1\right]\mu H_1(\mu) \quad , \quad \nu' \in (0,1) \quad ,$$

and integrate to obtain

$$-\frac{A(\nu_0)}{H_2(\nu_0)}\nu'\hat{q}_1W(\nu_0) + \frac{A(\nu')}{H_2(\nu')}N_1(\nu')H_1(\nu') - \nu'\hat{q}_1\overline{A} = -\frac{1}{H_2(-\nu_0)}\left[J(-\nu_0,\nu') - \nu'\hat{q}_1W(-\nu_0)\right] \quad . \tag{19}$$

Here,

$$N_1(\nu) = \nu\left\{\left[\lambda_1(\nu)\right]^2 + \left[\frac{c_1\nu\pi}{2}R_1(\nu,\nu)\right]^2\right\} \quad . \tag{20}$$

If now we rearrange Eq. (19) and multiply by $W(\nu')$, we can integrate to find

$$\overline{A} = -\frac{c_1K_1R_1(\nu_0,\nu_0)}{4H_1(\nu_0)H_2(-\nu_0)} + \frac{A(\nu_0)H_1(\nu_0)N_1(\nu_0)K_2}{\nu_0H_2(\nu_0)} \quad , \tag{21}$$

where the two constants K_1 and K_2 are given by

$$K_1 = \int_0^1 \frac{\nu W(\nu)(\nu_0 - \nu)}{N_1(\nu)H_1(\nu)(\nu_0 + \nu)}\,d\nu \tag{22a}$$

and

$$K_2 = \int_0^1 \frac{\nu W(\nu)}{N_1(\nu)H_1(\nu)}\,d\nu \quad . \tag{22b}$$

Equation (21) can be used in Eq. (15) to find $A(\nu_0)$, and subsequently $A(\nu)$ can be found from Eq. (19):

$$A(\nu_0) = -\frac{H_2(\nu_0)}{H_2(-\nu_0)} \times \frac{\left\{c_1\nu_0\left[1 - l_1\nu_0^2 + 2\nu_0\hat{c}_1 + \hat{q}_1K_1R_1(\nu_0,\nu_0)\right] - 4\nu_0H_1(\nu_0)\hat{q}_1W(-\nu_0)\right\}}{\left[4N_1(\nu_0)H_1(\nu_0)H_1(\nu_0)(1 - \hat{q}_1K_2) - 4\nu_0H_1(\nu_0)\hat{q}_1W(\nu_0)\right]} \tag{23}$$

and

$$A(\nu) = \frac{\nu H_2(\nu)}{N_1(\nu)H_1(\nu)}\left[\frac{A(\nu_0)H_1(\nu_0)N_1(\nu_0)}{\nu_0H_2(\nu_0)} - \frac{c_1(\nu_0 - \nu)R_1(\nu_0,\nu_0)}{4(\nu_0 + \nu)H_2(-\nu_0)H_1(\nu_0)}\right] \quad . \tag{24}$$

Equations (23) and (24) are explicit expressions for the expansion coefficients $A(\nu_0)$ and $A(\nu)$. Though our final results were obtained differently and are different in appearance from those of McCormick[4] and McCormick and Doyas,[5] they are similar in that there appear extra terms, in our case, W terms, in regard to either the isotropic scattering case, $l_1 = l_2 = 0$, or the single medium result, $c_2 = 0$.

We note that the neutron density in region 1 is given by

$$\rho_1(x) = \int_{-1}^1 \Psi_1(x,\mu)\,d\mu = A(\nu_0)\exp(-x/\nu_0) + \exp(x/\nu_0) + \int_0^1 A(\nu)\exp(-x/\nu)\,d\nu \quad . \tag{25}$$

Also, an asymptotic solution can be written as

$$\rho_{1a}(x) = A(\nu_0)\exp(-x/\nu_0) + \exp(x/\nu_0) \quad . \tag{26}$$

Thus, if we write

$$z_0 = -\frac{\nu_0}{2}\ln\left[-A(\nu_0)\right] \quad , \tag{27}$$

then Eq. (26) becomes

$$\rho_{1a}(x) = \exp(x/\nu_0) - \exp\left[-(x + 2z_0)/\nu_0\right] \quad . \tag{28}$$