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# Two-Group Neutron Transport Theory with Anisotropic Scattering

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ABSTRACT

*Solutions to the two-group neutron transport equation relevant to linearly anisotropic scattering are developed and full-range expansion and orthogonality theorems are discussed. In addition to the general forms given, explicit solutions are presented and all necessary normalization integrals are evaluated.*

## I. INTRODUCTION

The method of normal modes introduced by Case[1] has been used by several authors to study, with varying degrees of success, the two-group or multigroup version of the neutron transport equation in plane geometry. We should like to review briefly those papers most related to the current investigation. Żelazny and Kuszell[2] reported the first application of the singular eigenfunction expansion technique to the two-group model for isotropic scattering; however, in that early paper no explicit results were obtained and, in fact, it was not observed that full-range

[1] K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).
[2] R. ŻELAZNY and A. KUSZELL, *Ann. Phys.*, **16**, 81 (1961).

problems could be solved in closed form. Siewert and Zweifel[3,4] considered the picket-fence model in radiative transfer, and for this rather special case of the multigroup transport equation (the determinant of the transfer matrix is zero), they were able to prove rigorously both the full- and half-range completeness and orthogonality theorems. Closed-form results for several half-space problems were thus established.[3,4] Leonard and Ferziger[5] avoided the Fredholm equations reported by Żelazny and Kuszell[2] and thus were able to construct a more satisfactory proof of the relevant full-range completeness theorem; their "proof" of half-range completeness,[6] however, is not considered definitive because of the assumptions made regarding the partial indices essential to studies of systems of singular-integral equations.[7] Siewert and Shieh[8] proved explicitly the full-range completeness theorem for the two-group case of isotropic scattering, a result which was extended to the multigroup model by Yoshimura and Katsuragi.[9]

A relatively recent paper by Shultis[10] is more closely related to the present work. He considers the symmetric multigroup model with an arbitrary degree of anisotropic scattering. Though the model considered by Shultis is quite general, the reported analysis (in addition to being plagued by printing errors) is somewhat less complete than may be desired. The paper[10] suffers, as does Ref. 2, from the fact that the singular-integral equations considered in the "proof" of full-range completeness are reduced to Fredholm equations rather than being solved explicitly. In regard to half-range applications, Shultis'[10] "proof" of half-range completeness also is not considered definitive.[7] A study of the anisotropic scattering model in the multigroup formulation is currently being pursued by Silvennoinen and Zweifel.[11] Burniston and Siewert[12] have reported a proof of the half-range expansion theorem relevant to a related study of the scattering of polarized light.

In this paper, we wish to demonstrate that the full-range expansion theorem appropriate to the two-group neutron-transport equation with linearly anisotropic scattering can be solved without introducing cumbersome Fredholm equations. In fact, as for most cases,[13] the full-range expansion theorem proved here is established constructively, in that explicit closed-form results are obtained. Of course, a constructive proof of completeness is not absolutely essential since the orthogonality theorem and the required normalization integrals given in Sec. IV may be used conveniently to determine all expansion coefficients; however, valid proof of completeness is required, and we believe the proof given in Sec. III has been given sufficient attention to be considered definitive.

In addition to the general forms of the continuum normal modes used to prove the completeness theorem, specifically normalized solutions are given explicitly, and all normalization integrals related to the orthogonality theorem are evaluated. For a related problem,[14] the analogous full-range integrals have proved to be useful for half-space applications as well as for full-range expansions. Finally, to illustrate the established formalism, the infinite-medium Green's function is constructed.

## II. NORMAL MODES OF THE TRANSPORT EQUATION

We write the two-group neutron transport equation in plane geometry for linearly anisotropic scattering as

$$\mu \frac{\partial}{\partial x} \boldsymbol{\Psi}(x, \mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(x, \mu) = \boldsymbol{C} \int_{-1}^{1} \boldsymbol{\Psi}(x, \mu')d\mu' + \boldsymbol{B}\mu \int_{-1}^{1} \boldsymbol{\Psi}(x, \mu')\mu' d\mu' \quad . \tag{1}$$

Here $\boldsymbol{\Psi}(x, \mu)$ is a two-vector, the elements of which are the group angular fluxes, $\boldsymbol{C}$ and $\boldsymbol{B}$, with elements $c_{ij}$ and $b_{ij}$, the transfer matrices, and since we use the optical variable $x$, the $\boldsymbol{\Sigma}$ matrix is given by

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \sigma > 1 \quad . \tag{2}$$

In the usual manner,[13] we seek solutions of Eq. (1) of the form

$$\boldsymbol{\Psi}(x, \mu) = \boldsymbol{F}(\xi, \mu) \exp(-x/\xi) \tag{3}$$

and thus obtain

$$(\xi \boldsymbol{\Sigma} - \mu \boldsymbol{I})\boldsymbol{F}(\xi, \mu) = \xi \boldsymbol{\Delta}(\xi\mu)\boldsymbol{M}(\xi) \quad , \tag{4}$$

where $\boldsymbol{I}$ is the unit matrix, $\boldsymbol{\Delta}(x) = \boldsymbol{C} + x\boldsymbol{A}$, $\boldsymbol{A} = \boldsymbol{B}[\boldsymbol{\Sigma} - 2\boldsymbol{C}]$, and

$$\boldsymbol{M}(\xi) = \int_{-1}^{1} \boldsymbol{F}(\xi, \mu)d\mu \quad . \tag{5}$$

Considering first the discrete spectrum $\xi = \nu_i \notin [-1, 1]$, we find

$$\boldsymbol{F}(\pm\nu_i, \mu) = \nu_i \boldsymbol{D}(\nu_i, \pm\mu)\boldsymbol{\Delta}(\pm\nu_i\mu)\boldsymbol{M}(\nu_i) \quad , \tag{6}$$

where $\pm\nu_i$ are the zeros of the dispersion function $\Lambda(z) = \det \boldsymbol{\Lambda}(z)$, with

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} - z \int_{-1}^{1} \boldsymbol{D}(z, \mu)[\boldsymbol{C} + \mu^2 \boldsymbol{\Sigma}^{-1}\boldsymbol{A}]d\mu \quad , \tag{7a}$$

or alternatively

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + z \int_{-1}^{1} \boldsymbol{\Psi}(\mu) \frac{d\mu}{\mu - z} \quad , \tag{7b}$$

and $\boldsymbol{M}(\nu_i)$ is a null vector of $\boldsymbol{\Lambda}(\nu_i)$:

$$\boldsymbol{\Lambda}(\nu_i)\boldsymbol{M}(\nu_i) = \boldsymbol{0} \quad . \tag{8}$$

In addition, $\boldsymbol{\Psi}(\mu)$ is the characteristic matrix

$$\boldsymbol{\Psi}(\mu) = \begin{bmatrix} \Theta(\mu) & 0 \\ 0 & 1 \end{bmatrix} [\boldsymbol{C} + \mu^2 \boldsymbol{\Sigma}\boldsymbol{A}] \triangleq \boldsymbol{\Theta}(\mu)[\boldsymbol{C} + \mu^2 \boldsymbol{\Sigma}\boldsymbol{A}] \quad , \tag{9}$$

[3] C. E. SIEWERT and P. F. ZWEIFEL, *Ann. Phys.*, **36**, 61 (1966).
[4] C. E. SIEWERT and P. F. ZWEIFEL, *J. Math. Phys.*, **7**, 2092 (1966).
[5] A. LEONARD and J. H. FERZIGER, *Nucl. Sci. Eng.*, **26**, 170 (1966).
[6] A. LEONARD and J. H. FERZIGER, *Nucl. Sci. Eng.*, **26**, 181 (1966).
[7] E. E. BURNISTON, C. E. SIEWERT, P. SILVENNOINEN, and P. F. ZWEIFEL, *Nucl. Sci. Eng.*, **45**, 331 (1971).
[8] C. E. SIEWERT and P. S. SHIEH, *J. Nucl. Eng.*, **21**, 383 (1967).
[9] T. YOSHIMURA and S. KATSURAGI, *Nucl. Sci. Eng.*, **33**, 297 (1968).
[10] J. K. SHULTIS, *Nucl. Sci. Eng.*, **38**, 83 (1969).
[11] P. SILVENNOINEN and P. F. ZWEIFEL, Personal Communication.
[12] E. E. BURNISTON and C. E. SIEWERT, *J. Math. Phys.*, **11**, 3416 (1970).
[13] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Company, Reading, Massachusetts (1967).
[14] C. E. SIEWERT, *J. Quant. Spectrosc. Radiat. Transfer*, (in press).

with

$$\Theta(\mu) = 1 \quad , \quad \mu \epsilon \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right) \quad ,$$
$$= 0 \quad , \quad \text{otherwise} \quad ,$$

and

$$D(z, \mu) = \begin{bmatrix} \frac{1}{\sigma z - \mu} & 0 \\ 0 & \frac{1}{z - \mu} \end{bmatrix} . \tag{10}$$

Note that $\Lambda(z) = \Lambda(-z)$ so that the zeros of $\Lambda(z)$ occur in ± pairs. We denote the total number of these discrete eigenvalues by $2\kappa$. We prefer to devote Sec. IV to more explicit solutions and thus will not write out Eq. (6) for a particular choice of $M(\nu_i)$. However, we integrate Eq. (7b) to obtain the dispersion function

$$\Lambda(z) = 1 + 2z^2 P_1(z) - 2zP_2(z) \tanh^{-1} \frac{1}{\sigma z} - 2zP_3(z) \tanh^{-1} \frac{1}{z}$$
$$+ 4z^2 P_4(z) \tanh^{-1} \frac{1}{\sigma z} \tanh^{-1} \frac{1}{z} \quad , \tag{11}$$

where the four polynomials are given by

$$P_1(z) = a_{11} + a_{22} + 2Az^2 \quad , \tag{12a}$$

$$P_2(z) = c_{11} + (\sigma a_{11} + 2c_{11}a_{22} - 2c_{12}a_{21})z^2 + 2\sigma Az^4 \quad , \tag{12b}$$

$$P_3(z) = c_{22} + (a_{22} + 2c_{22}a_{11} - 2c_{21}a_{12})z^2 + 2Az^4 \quad , \tag{12c}$$

$$P_4(z) = C + (\sigma c_{22}a_{11} + c_{11}a_{22} - c_{12}a_{21} - \sigma c_{21}a_{12})z^2 + \sigma Az^4 \quad . \tag{12d}$$

We use here the abbreviations $C$ and $A$ for det $C$ and det $A$ and denote the elements of $A$ by $a_{ij}$.

Note that the upper row of $\Lambda(z)$ has a branch cut $\left[-\frac{1}{\sigma}, \frac{1}{\sigma}\right]$, whereas the lower row has a branch cut $[-1, 1]$. The function $\Lambda(z)$ is otherwise analytic in the complex plane and is bounded at infinity:

$$\Lambda(\infty) = I - 2\Sigma^{-1}\left[C + \frac{1}{3}\Sigma^{-1}A\right] \quad . \tag{13}$$

In addition, the Plemelj formulas[15] relate the boundary values of $\Lambda(z)$ as the branch cut is approached from above (+) and below (-):

$$\Lambda^+(\nu) - \Lambda^-(\nu) = 2\pi i\nu \Psi(\nu) \tag{14a}$$

and

$$\Lambda^+(\nu) + \Lambda^-(\nu) = 2\lambda(\nu) \quad , \tag{14b}$$

where

$$\lambda(\nu) = I + \nu P \int_{-1}^{1} \Psi(\mu) \frac{d\mu}{\mu - \nu} \quad , \tag{15}$$

with the symbol $P$ denoting that the integral is to be evaluated in the Cauchy principal-value sense.

For the continuum $\xi = \nu \epsilon (-1, 1)$, we express the solution to Eq. (4) as

$$F(\nu, \mu) = [\nu K(\nu, \mu) + \omega(\nu)\delta(\nu, \mu)]\Delta(\nu\mu)M(\nu) \quad , \tag{16}$$

where

$$K(\nu, \mu) = \begin{bmatrix} \frac{P}{\sigma\nu - \mu} & 0 \\ 0 & \frac{P}{\nu - \mu} \end{bmatrix} \tag{17a}$$

and

$$\delta(\nu, \mu) = \begin{bmatrix} \delta(\sigma\nu - \mu) & 0 \\ 0 & \delta(\nu - \mu) \end{bmatrix} . \tag{17b}$$

Equation (16) may now be integrated over $\mu$ from -1 to 1 to yield

$$[\lambda(\nu) - \omega(\nu)\Psi(\nu)]M(\nu) = 0 \quad , \tag{18}$$

and hence the function $\omega(\nu)$ is determined from the requirement

$$\det[\lambda(\nu) - \omega(\nu)\Psi(\nu)] = 0 \quad . \tag{19}$$

We should now like to label the following two regions of interest:

$$\text{Region } ① \Rightarrow \nu \epsilon \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right)$$

$$\text{Region } ② \Rightarrow \nu \epsilon \left(-1, -\frac{1}{\sigma}\right) \quad \text{or} \quad \left(\frac{1}{\sigma}, 1\right) \quad .$$

In general, for $\nu \epsilon$ region ①, Eq. (19) is quadratic in $\omega(\nu)$ and thus yields two independent solutions $\omega_1^{(1)}(\nu)$ and $\omega_2^{(1)}(\nu)$; however, for $\nu \epsilon$ region ②, Eq. (19) is simply linear in $\omega(\nu)$, and therefore only one solution $\omega^{(2)}(\nu)$ is available. Thus, for $\nu \epsilon$ region ① we find the two solutions

$$F_\alpha^{(1)}(\nu, \mu) = [\nu K(\nu, \mu) + \omega_\alpha^{(1)}(\nu)\delta(\nu, \mu)]\Delta(\nu\mu)M_\alpha^{(1)}(\nu) \quad ,$$
$$\nu \epsilon \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right) \quad , \quad \alpha = 1 \text{ and } 2 \quad , \tag{20a}$$

whereas for $\nu \epsilon$ region ② there is only one solution

$$F^{(2)}(\nu, \mu) = [\nu K(\nu, \mu) + \omega^{(2)}(\nu)\delta(\nu, \mu)]\Delta(\nu\mu)M^{(2)}(\nu) \quad ,$$
$$\nu \epsilon \left(-1, -\frac{1}{\sigma}\right) \quad \text{or} \quad \left(\frac{1}{\sigma}, 1\right) \quad . \tag{20b}$$

In Sec. IV we give more explicit forms for related continuum solutions. Here our general solution to Eq. (1) is written as

$$\Psi(x, \mu) = \sum_{i=1}^{\kappa} [A(\nu_i)F(\nu_i, \mu) \exp(-x/\nu_i) + A(-\nu_i)F(-\nu_i, \mu)$$
$$\times \exp(x/\nu_i)] + \int_{①} [A_1^{(1)}(\nu)F_1^{(1)}(\nu, \mu)$$
$$+ A_2^{(1)}(\nu)F_2^{(1)}(\nu, \mu)] \exp(-x/\nu)\, d\nu$$
$$+ \int_{②} A^{(2)}(\nu)F^{(2)}(\nu, \mu) \exp(-x/\nu)\, d\nu \quad . \tag{21}$$

To complete this section we should like to introduce the adjoint equation

$$\mu \frac{\partial}{\partial x} \Psi_a(x, \mu) + \Sigma\Psi_a(x, \mu) = \tilde{C} \int_{-1}^{1} \Psi_a(x, \mu')\, d\mu'$$
$$+ \tilde{B}\mu \int_{-1}^{1} \Psi_a(x, \mu')\mu'\, d\mu' \quad , \tag{22}$$

where the superscript tilde is used to denote the transpose operation. Again proposing solutions of the form

$$\Psi_a(x, \mu) = F_a(\xi, \mu) \exp(-x/\xi) \quad , \tag{23}$$

[15] N. I. MUSKHELISHVILI, *Singular Integral Equations*, P. Noordhoff, Groningen, Holland (1953).

we find a set of solutions $F_a(\pm\nu_i, \mu)$, $F_{a1}^{(1)}(\nu, \mu)$, $F_{a2}^{(1)}(\nu, \mu)$, and $F_a^{(2)}(\nu, \mu)$ analogous to Eqs. (6) and (20) with, however, $c_{ij}$ and $b_{ij}$ replaced by $c_{ji}$ and $b_{ji}$. We note the dispersion matrix for this adjoint case is

$$\mathbf{\Lambda}_a(z) = I + z \int_{-1}^{1} \mathbf{\Psi}_a(\mu) \frac{d\mu}{\mu - z} \quad , \tag{24}$$

where

$$\mathbf{\Psi}_a(\mu) = \mathbf{\Theta}(\mu)[\widetilde{C} + \mu^2 \mathbf{\Sigma} A_a] \quad , \tag{25}$$

with $A_a = \widetilde{B}[\mathbf{\Sigma} - 2\widetilde{C}]$. Silvennoinen and Zweifel[16] have shown that the direct and adjoint spectra are identical, as can easily be verified for the model considered here. Thus

$$\Lambda_a(z) = \det \mathbf{\Lambda}_a(z) = \Lambda(z) = \det \mathbf{\Lambda}(z) \quad . \tag{26}$$

In addition, it is a simple matter to establish the following relationship between the direct and adjoint dispersion matrices

$$[\widetilde{C} + z^2 \widetilde{\mathbf{\Lambda}} \mathbf{\Sigma}]\mathbf{\Lambda}_a(z) = \widetilde{\mathbf{\Lambda}}(z)[\widetilde{C} + z^2 \mathbf{\Sigma} A_a] \quad . \tag{27}$$

## III. THE FULL-RANGE EXPANSION THEOREM

Having established the elementary solutions of Eq. (1), we now show that these normal modes form a complete basis set for the expansion of arbitrary two-vector Hölder functions.[15]

*Theorem I: The functions* $F(\pm\nu_i, \mu)$, $i = 1, 2, \ldots, \kappa$, $F_1^{(1)}(\nu, \mu)$ *and* $F_2^{(1)}(\nu, \mu)$, $\nu \epsilon \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right)$, *and* $F^{(2)}(\nu, \mu)$, $\nu\epsilon\left(-1, -\frac{1}{\sigma}\right)$ *and* $\left(\frac{1}{\sigma}, 1\right)$, *form a complete basis for the expansion of arbitrary two-vector Hölder functions* $\mathbf{\Phi}(\mu)$ *defined on the full-range* $\mu\epsilon(-1, 1)$ *in the sense that*

$$\begin{aligned}\mathbf{\Phi}(\mu) = &\sum_{i=1}^{\kappa} [A(\nu_i)F(\nu_i, \mu) + A(-\nu_i)F(-\nu_i, \mu)] \\ &+ \int_{①} [A_1^{(1)}(\nu)F_1^{(1)}(\nu, \mu) + A_2^{(1)}(\nu)F_2^{(1)}(\nu, \mu)]\, d\nu \\ &+ \int_{②} A^{(2)}(\nu)F^{(2)}(\nu, \mu)\, d\nu \quad , \quad \mu\epsilon(-1, 1) \quad .\end{aligned} \tag{28}$$

To prove the theorem, we construct an analytical solution to the coupled singular-integral equations above. If we enter Eqs. (20) into Eq. (28) and make use of Eq. (18), we find

$$\mathbf{\Phi}'(\mu) = \int_{-1}^{1} [\nu K(\nu, \mu)\mathbf{\Delta}(\nu\mu) + \boldsymbol{\delta}(\nu, \mu)\boldsymbol{\lambda}(\nu)]A(\nu)\, d\nu \quad , \tag{29}$$

where for the sake of brevity we have defined

$$\mathbf{\Phi}'(\mu) = \mathbf{\Phi}(\mu) - \sum_{i=1}^{\kappa} [A(\nu_i)F(\nu_i, \mu) + A(-\nu_i)F(-\nu_i, \mu)] \tag{30}$$

and introduced the vector

$$\begin{aligned}A(\nu) = &[A_1^{(1)}(\nu)M_1^{(1)}(\nu) + A_2^{(1)}(\nu)M_2^{(1)}(\nu)]\Theta(\nu) \\ &+ A^{(2)}(\nu)M^{(2)}(\nu)[1 - \Theta(\nu)] \quad , \quad \nu\epsilon(-1, 1) \quad .\end{aligned} \tag{31}$$

If we now change $\mu$ to $\sigma\mu$ in the top row of Eq. (29), premultiply by $\mathbf{\Sigma}$, and integrate the $\boldsymbol{\delta}$ term, we obtain

$$\begin{aligned}\mathbf{\Sigma}\,\mathbf{\Phi}''(\mu) = &\boldsymbol{\lambda}(\mu)A(\mu) + \mathbf{\Theta}(\mu)P\int_{-1}^{1} \nu[C + \nu\mu\mathbf{\Sigma}A]A(\nu) \frac{d\nu}{\nu - \mu} \quad , \\ &\mu\epsilon(-1, 1) \quad ,\end{aligned} \tag{32}$$

where

$$\mathbf{\Phi}''(\mu) = \mathbf{\Theta}(\mu)\begin{bmatrix}\phi_1^1(\sigma\mu) \\ \phi_2^1(\mu)\end{bmatrix} \quad . \tag{33}$$

Equation (32) is now quite concise and may be solved using the standard techniques of Muskhelishvili.[15] We therefore introduce the sectionally holomorphic function

$$N(z) = \frac{1}{2\pi i} \int_{-1}^{1} \nu[C + \nu z \mathbf{\Sigma} A]A(\nu) \frac{d\nu}{\nu - z} \tag{34}$$

with boundary values satisfying

$$\pi i[N^+(\mu) + N^-(\mu)] = P\int_{-1}^{1} \nu[C + \nu\mu\mathbf{\Sigma}A]A(\nu) \frac{d\nu}{\nu - \mu} \tag{35a}$$

and

$$N^+(\mu) - N^-(\mu) = \mu[C + \mu^2 \mathbf{\Sigma} A]A(\mu) \quad . \tag{35b}$$

Equation (32) can now be premultiplied by $\mu[C + \mu^2 \widetilde{A}_a \mathbf{\Sigma}]$ to yield, after Eqs. (14), (27), and (35) are invoked,

$$\begin{aligned}\mu[C + \mu^2 \widetilde{A}_a \mathbf{\Sigma}]\mathbf{\Sigma}\,\mathbf{\Phi}''(\mu) = &\widetilde{\mathbf{\Lambda}}_a^+(\mu)N^+(\mu) - \widetilde{\mathbf{\Lambda}}_a^-(\mu)N^-(\mu) \quad , \\ &\mu\epsilon(-1, 1) \quad ,\end{aligned} \tag{36}$$

which clearly is a matrix version of a special case of the inhomogeneous Riemann-Hilbert problem.[15] Note, therefore, that the vector

$$P(z) = \widetilde{\mathbf{\Lambda}}_a(z)N(z) - \frac{1}{2\pi i}\int_{-1}^{1} \mu[C + \mu z \widetilde{A}_a \mathbf{\Sigma}]\mathbf{\Sigma}\,\mathbf{\Phi}''(\mu) \frac{d\mu}{\mu - z} \tag{37}$$

is analytic in the complex plane cut from -1 to 1 along the real axis. Further, since Eq. (36) ensures that $P(z)$ is continuous across the cut, $P(z)$ must be an entire function. Observing Eq. (34), we write

$$N(\infty) = -\frac{1}{2\pi i} \mathbf{\Sigma} A \int_{-1}^{1} A(\nu)\nu^2 d\nu \quad , \tag{38}$$

whereas Eq. (24) yields

$$\widetilde{\mathbf{\Lambda}}_a(\infty) = I - 2\left[C + \frac{1}{3}\widetilde{A}_a \mathbf{\Sigma}^{-1}\right]\mathbf{\Sigma}^{-1} \quad . \tag{39}$$

If Eq. (4) is now multiplied by $\mu$ and integrated over $\mu$ from -1 to 1, there results

$$\int_{-1}^{1} F(\xi, \mu)\mu^2 d\mu = \xi^2\left[\mathbf{\Sigma} - \frac{2}{3}B\right][\mathbf{\Sigma} - 2C]M(\xi) \tag{40}$$

for all appropriate $\xi$. Equation (29) may now be multiplied by $\mu^2$ and integrated over all $\mu$ to yield, after use is made of Eq. (40),

$$\int_{-1}^{1} \mathbf{\Phi}'(\mu)\mu^2 d\mu = \left[\mathbf{\Sigma} - \frac{2}{3}B\right][\mathbf{\Sigma} - 2C]\int_{-1}^{1} A(\nu)\nu^2 d\nu \quad . \tag{41}$$

Equations (38), (39), and (41) may be used in Eq. (37) to show that

$$P(\infty) = \mathbf{0} \quad , \tag{42}$$

and thus $P(z) \equiv \mathbf{0}$. The vector $N(z)$ is therefore established:

$$N(z) = \widetilde{\mathbf{\Lambda}}_a^{-1}(z) \frac{1}{2\pi i}\int_{-1}^{1} \mu[C + \mu z \widetilde{A}_a \mathbf{\Sigma}]\mathbf{\Sigma}\,\mathbf{\Phi}''(\mu) \frac{d\mu}{\mu - z} \quad , \tag{43}$$

or alternatively

$$N(z) = -\widetilde{\mathbf{\Lambda}}_a^{-1}(z) \frac{1}{2\pi i}\int_{-1}^{1} \mu[C + \mu z \widetilde{A}_a]D(z, \mu)\mathbf{\Phi}'(\mu)\, d\mu \quad . \tag{44}$$

For arbitrary $\mathbf{\Phi}'(\mu)$, Eq. (44) yields a meromorphic vector $N(z)$ since $\det \mathbf{\Lambda}_a(z)$ has zeros at $z = \pm\nu_i$, $i = 1, 2, \ldots, \kappa$. Clearly then, Eq. (44) does not prescribe the correct $N(z)$

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[16]P. SILVENNOINEN and P. F. ZWEIFEL, *Nucl. Sci. Eng.*, **42**, 103 (1970).