MA 241 Practice on Series (page 1)

1. a) $3 + \frac{3}{4} + \cdots + \frac{3}{4^{n-1}} + \cdots = \sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^{n-1}$ $a = 3$, $r = 1/4$

Converges b/c geometric w/ $|r| = 1/4 < 1$.

$\text{Sum} = \frac{a}{1-r} = \frac{3}{1-1/4} = \frac{3}{3/4} = \underline{\underline{4}}$

b) $\sum_{n=1}^{\infty} 2^{-n} 3^{n-1} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{2 \cdot 2^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$ $a = 1/2$, $r = 3/2$

diverges b/c geometric w/ $|r| = 3/2 > 1$.

2. a) $\sum_{n=1}^{\infty} \frac{3n}{5n-1}$ $\lim_{n\to\infty} \frac{3n}{5n-1} = \frac{3}{5} \neq 0$ $\therefore$ diverges (Divergence test)

b) $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ $\lim_{n\to\infty} \frac{1}{n^2+3} = 0$ $\therefore$ Div. Test inconclusive.

use comparison test $\rightarrow$ compare w/ $\frac{1}{n^2}$ (converging p-series)

$n^2+3 > n^2 \Rightarrow \frac{1}{n^2} > \frac{1}{n^2+3}$

Since $\sum \frac{1}{n^2}$ converges (p-series w/ $p=2$) and $\frac{1}{n^2} > \frac{1}{n^2+3}$, $\sum \frac{1}{n^2+3}$ converges by the comparison test.

3. $\sum_{n=1}^{\infty} \frac{5}{(n+2)(n+3)}$ $\frac{5}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$ (partial fractions to decompose telescoping series)

$5 = A(n+3) + B(n+2)$

$A + B = 0$

$3A + 2B = 5$ $\rightarrow \boxed{A = 5, B = -5}$

So our series is

$\sum_{n=1}^{\infty} \left(\frac{5}{n+2} - \frac{5}{n+3}\right) \rightarrow$ n^{th} partial sum $S_n = \underbrace{\left(\frac{5}{3} - \cancel{\frac{5}{4}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{5}{4}} - \cancel{\frac{5}{5}}\right)}_{n=2} + \cdots + \left(\cancel{\frac{5}{n+2}} - \frac{5}{n+3}\right)$

$S = \lim_{n\to\infty} \left(\frac{5}{3} - \frac{5}{n+3}\right) = \frac{5}{3}$ converges. sum $= 5/3$.

4. a) $\sum_{n=1}^{\infty} \frac{1}{n^6}$ $p = 6 > 1$ $\therefore$ converges

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ $p = \frac{2}{3} \le 1$ $\therefore$ diverges

5. a) $\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$ $f(x) = \frac{1}{(3+2x)^2}$ is positive, decreasing, and continuous (since $x \ge 1$)

[verify decreasing w/ derivative]

$$\lim_{b\to\infty} \int_1^b (3+2x)^{-2}\,dx$$

$$= \lim_{b\to\infty} \left. \frac{-(3+2x)^{-1}}{2} \right|_1^b = \lim_{b\to\infty} \left[\frac{-1}{2(3+2b)}^{\nearrow 0} + \frac{1}{2(3+2)} \right] = \frac{1}{10}$$ nonzero & finite

the integral converges $\therefore$ the series converges.

b) $\sum_{n=1}^{\infty} \frac{1}{4n+7}$ $f(x) = \frac{1}{4x+7}$ is positive, decreasing, continuous (because $x \ge 1$)

[again - verify these]

$$\lim_{b\to\infty} \int_1^b \frac{1}{4x+7}\,dx = \lim_{b\to\infty} \left. \frac{\ln|4x+7|}{4} \right|_1^b = \lim_{b\to\infty} \left[\frac{\ln|4b+7|}{4} - \frac{\ln|4+7|}{4} \right] = \infty$$

the integral diverges $\therefore$ the series diverges.

6. a) $\sum_{n=1}^{\infty} \frac{1}{n^4+n^2+1}$ compare w/ $\sum_{n=1}^{\infty} \frac{1}{n^4}$ (converging p-series)

$n^4+n^2+1 > n^4$ (since $n \ge 1$)

$\Rightarrow \frac{1}{n^4} > \frac{1}{n^4+n^2+1}$ $\therefore$ converges by comparison test

b) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ compare w/ $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$ (converging geometric series)

$n3^n \ge 3^n$ for $n \ge 1$

$\Rightarrow \frac{1}{3^n} \ge \frac{1}{n3^n}$ $\therefore$ converges by comparison test

7. a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$ appears to behave like $\frac{1}{\sqrt{n}}$ for large n (diverging p-series)

try limit comparison

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+4}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+4} = 1 > 0 \text{ and finite}$$

$\therefore$ diverges by limit comparison b/c $\frac{1}{\sqrt{n}}$ diverges.

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^3-5n}}$ appears to behave like $\frac{1}{2n^{3/2}}$ for large n (converging p-series)

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{4n^3-5n}}}{\frac{1}{2n^{3/2}}} = \lim_{n\to\infty} \frac{2n^{3/2} / n^{3/2}}{\sqrt{(4n^3-5n)/n^3}} = \lim_{n\to\infty} \frac{2}{\sqrt{4-5/n^2}} = 1$$

$\therefore$ converges by lim. comp. b/c $\frac{1}{n^{3/2}}$ converges.

8. a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ $\lim_{n\to\infty} \frac{1}{2n-1} = 0$

decreasing: $2(n+1)-1 > 2n-1 \rightarrow \frac{1}{2n-1} > \frac{1}{2(n+1)-1}$

✓ converges by alternating series test.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$ $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

instead, look at $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1} \rightarrow$ limit dne. $\therefore$ diverges by Div. Test.

9. a) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\ \cancel{3^n}}{\cancel{3^n}\cdot 3\ \cancel{n!}} \right| = \infty$

diverges by Ratio Test

b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ $\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2^n}\cdot 2 \cdot n^2}{(n+1)^2\ \cancel{2^n}} \right| = 2\left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 = 2 > 1$

$\therefore$ diverges by Ratio Test