W-Constraints for Simple Singularities

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Outline

Gromov–Witten Theory

- **Gromov–Witten invariants** of a manifold $X \Leftrightarrow$ generating series $\mathcal{D}_X$ – total descendant potential.
- **Witten–Kontsevich** function $\mathcal{D}_{\text{pt}} \Leftrightarrow$ intersection theory on $\overline{\mathcal{M}}_{g,n}$.
  - τ-function for the KdV hierarchy.
  - Virasoro constraints $L_n \mathcal{D}_{\text{pt}} = 0$ $(n \geq -1)$. String eq. $L_{-1}\mathcal{D}_{\text{pt}} = 0$.
- **Genus 0** Gromov–Witten invariants of $X \Rightarrow$ quantum cup product on $H^*(X) \Rightarrow$ Frobenius algebra + flat connection $\Rightarrow$ Frobenius manifold.
- **Givental's formula** expresses $\mathcal{D}_X$ in terms of $\mathcal{D}_{\text{pt}}$ and the semisimple Frobenius structure.
  - Virasoro constraints for $\mathcal{D}_{\text{pt}} \Rightarrow$ Virasoro constraints for $\mathcal{D}_X$.

Main Questions

Questions

- Does $\mathcal{D}_X$ satisfy an integrable hierarchy?
- Does $\mathcal{D}_X$ satisfy Virasoro or W-constraints?

Generalized Witten Conjecture:

- The total descendant potential for h-spin curves is a τ-function for the h-th Gelfand–Dickey hierarchy and satisfies $\mathcal{W}_h$-constraints.
  - $\mathcal{W}_h = \mathcal{W}(2, 3, \ldots, h)$ is the Zamolodchikov–Fateev–Lukyanov W-algebra.
  - $\mathcal{W}_2 =$ Virasoro algebra.

Previous Work

- **Adler–van Moerbeke:** There is a unique τ-function for the h-th Gelfand–Dickey hierarchy solving the string equation. It satisfies the $\mathcal{W}_h$-constraints.
- **K. Saito:** Singularity theory $\Rightarrow$ semisimple Frobenius structure $\Rightarrow$ total descendant potential $\mathcal{D}_{X_N}$ defined by Givental's formula.
  - X_N is the type of the singularity.
  - $X_N = A_N, D_N, E_6, E_7$ or E_8 for simple singularities.
- **Fan–Jarvis–Ruan:** $\mathcal{D}_{A_N}$ is the total descendant potential for Gromov–Witten invariants of h-spin curves ($h = N + 1$).
  - Generalization for types D and E.
- **E. Frenkel–Givental–Milanov:** $\mathcal{D}_{X_N}$ ($X = A, D, E$) is a τ-function for the Kac–Wakimoto hierarchy of type X_N in the principal realization.
  - Type $A_N \Leftrightarrow h$-th Gelfand–Dickey hierarchy ($h = N + 1$).
  - This implies the generalized Witten conjecture.

Main Result

Theorem

The total descendant potential $\mathcal{D}_{X_N}$ $(X = A, D, E)$ *of a simple singularity satisfies* $\mathcal{W}_{X_N}$*-constraints.*

Remarks

- We can prove both the integrable hierarchy and W-constraints.
- The proof works more generally for weighted homogeneous singularities.
- The proof uses singularity theory, Givental's formula, and vertex algebras.

Simple Singularities

- $f(x_1, x_2, x_3)$ – homogeneous polynomial of degree h ($\deg x_i = a_i$) with an isolated critical point at $x = 0$ $\left(\frac{\partial f}{\partial x_i}(0) = 0\right)$.

Type	$\boldsymbol{h}$	$\boldsymbol{a_1, a_2, a_3}$	$\boldsymbol{f(x)}$	**Exponents**
A_N	$N+1$	$1, a, N+1-a$	$x_1^{N+1}+x_2x_3$	$1, 2, \dots, N$
D_N	$2N-2$	$2, N-2, N-1$	$x_1^{N-1}+x_1x_2^2+x_3^2$	$1, 3, \dots, 2N-3, N-1$
E_6	12	$3, 4, 6$	$x_1^4+x_2^3+x_3^2$	$1, 4, 5, 7, 8, 11$
E_7	18	$4, 6, 9$	$x_1^3x_2+x_2^3+x_3^2$	$1, 5, 7, 9, 11, 13, 17$
E_8	30	$6, 10, 15$	$x_1^5+x_2^3+x_3^2$	$1, 7, 11, 13, 17, 19, 23, 29$

- The exponents $m_1, \dots, m_N$ satisfy

$$q^{m_1} + \cdots + q^{m_N} = \frac{(q^h - q^{a_1})(q^h - q^{a_2})(q^h - q^{a_3})}{q^h(q^{a_1} - 1)(q^{a_2} - 1)(q^{a_3} - 1)} .$$

Milnor Fibration

- **Miniversal deformations:** $f_t(x) = f(x) + \sum_{i=1}^{N} t_i g_i(x)$.
  - $t = (t_1, \dots, t_N) \in \mathcal{T} = \mathbb{C}^N$.
  - $g_i(x)$ – homogeneous polynomials giving a basis of $\mathbb{C}[x]/(\partial f(x))$.
- **Milnor fibration:** $\mathcal{T} \times \mathbb{C}^3 \to \mathcal{T} \times \mathbb{C}, \;\; (t, x) \mapsto (t, f_t(x))$.
  - Fibers X_s ($s = (t, \lambda)$) consist of $(t, x) \in \mathcal{T} \times \mathbb{C}^3$ such that $f_t(x) = \lambda$.
  - Smooth fibration outside the discriminant Σ consisting of s such that X_s is singular.
  - All smooth fibers are diffeomorphic to X_1, $1 = (0, 1) \in \mathcal{T} \times \mathbb{C}$.
- **Milnor lattice:** $Q = H_2(X_1; \mathbb{Z})$ with negative the intersection form $\cong$ root lattice of type X_N.
  - Homomorphism $Q \to H_2(X_s; \mathbb{Z})$ for generic $s \in \Sigma$ and every path from 1 to s avoiding Σ.
  - $\alpha \in Q$ is a vanishing cycle if $\alpha \mapsto 0$ and $(\alpha|\alpha) = 2$.
  - The set R of vanishing cycles is a root system of type X_N.

Monodromy Representation

- **Fundamental group** of $(\mathcal{T}\times\mathbb{C})\setminus\Sigma$ acts on Q by orthogonal transformations. Image $=$ Weyl group W of type X_N.
- **Picard–Lefschetz:** Small loop around $s\in\Sigma\Rightarrow$ reflection $r_\alpha\in W$ $(\alpha\in R$ vanishing over $s)$.
    - $r_\alpha(\beta)=\beta-(\alpha|\beta)\alpha$.
    - W is generated by r_α $(\alpha\in R)$.
- **Classical monodromy:** Big loop around $\Sigma\Rightarrow$ Coxeter element $\sigma\in W$.
    - $\sigma=r_{\alpha_1}\cdots r_{\alpha_N}$ where $\alpha_1,\dots,\alpha_N$ is a basis of simple roots.
    - $|\sigma|=h$ is the Coxeter number.
    - σ is diagonalizable on $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}Q$ with eigenvalues $e^{2\pi\mathrm{i}m_k/h}$.
    - σ has no fixed points in $\mathfrak{h}$.
- **Generalized root system (K. Saito):** A lattice Q with a subset of roots R invariant under the action of the reflection group W generated by r_α $(\alpha\in R)$.

Affine Lie Algebras

- $\mathfrak{g}$ – finite-dimensional Lie algebra with a symmetric invariant bilinear form $(\cdot|\cdot)$.
  - E.g., Killing form, or $(a|b) = \mathrm{tr}(ab)$ for $\mathfrak{g} \subseteq \mathfrak{gl}_n$.
- **Affine Lie algebra** $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$.
  - $[a_m, b_n] = [a, b]_{m+n} + m\delta_{m,-n}(a|b)K$, where $a_m = at^m$.
  - K is central.
- **Verma module** $M(\Lambda_0) = \mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}$ of level 1 $(K = 1)$.
- **Basic representation** $V(\Lambda_0)$ – irreducible quotient of $M(\Lambda_0)$.
  - $V(\Lambda_0)$ has the structure of a vertex algebra.

Definition (Borcherds)

A vertex algebra is a vector space V (space of states) with a vacuum vector $1 \in V$ and state-field correspondence $Y(\cdot, z)\cdot\colon V \otimes V \to V((z))$ + axioms.

Lattice Vertex Algebras

Notation

- Q – integral (even) lattice.
- $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ – vector space with symmetric bilinear form $(\cdot|\cdot)$.
- $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ – Heisenberg Lie algebra.
- $B = M(\Lambda_0) \cong S(\mathfrak{h}[t^{-1}]t^{-1})$ – (bosonic) Fock space.
- $\mathbb{C}_\varepsilon[Q] = \mathrm{span}_{\mathbb{C}}\{e^\alpha | \alpha \in Q\}$ – ε-twisted group algebra.
  - $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$.
  - $\varepsilon \colon Q \times Q \to \{\pm 1\}$ – bimultiplicative; $\varepsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2}$.

Definition (Borcherds)

Lattice vertex algebra $V_Q = B \otimes \mathbb{C}_\varepsilon[Q]$.

- $Y(a_{-1}1, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ for $a \in \mathfrak{h}$, $a_n = at^n$.
- $Y(e^\alpha, z) = e^\alpha z^{\alpha_0} \exp\left(-\sum_{n<0} \alpha_n \frac{z^{-n}}{n}\right) \exp\left(-\sum_{n>0} \alpha_n \frac{z^{-n}}{n}\right)$.

The Vertex Algebra $\mathcal{W}_{X_N}$

- **I. Frenkel–Kac:** When Q is a root lattice of type X_N ($X = A, D, E$), $V_Q \cong V(\Lambda_0)$ – basic representation of $\hat{\mathfrak{g}}$.
  - $\mathfrak{g}$ acts by derivations on V_Q via $a \mapsto a_0$.
  - Screening operators $(e^\alpha)_0 = \mathrm{Res}_{z=0}\, Y(e^\alpha, z)$ are derivations of V_Q.
- **E. Frenkel–Kac–Radul–Wang:** The W-algebra $\mathcal{W}_{X_N}$ (with central charge N) $\cong$ kernel of screening operators ($B \to V_Q$) $\cong$ $\mathfrak{g}$-invariant part of $V(\Lambda_0)$.
  - $\mathcal{W}_{X_N} \subset B$ is fixed pointwise by the Weyl group W.
  - $\mathcal{W}_{X_N}$ is freely generated by fields of conformal weights $m_1 + 1, \ldots, m_N + 1$ (**Feigin–E. Frenkel**).
  - $\mathcal{W}_{X_N}$ contains a Virasoro field; $m_1 + 1 = 2$.
  - $\mathcal{W}_{A_N} = \mathcal{W}_{N+1}$ (central charge N); freely generated by fields of conformal weights $2, \ldots, N+1$.
  - $\mathcal{W}_{A_1} =$ Virasoro vertex algebra (central charge 1).

Main Results

Theorem 1

The total descendant potential $\mathcal{D}_{X_N}$ $(X = A, D, E)$ belongs to a σ-twisted representation of the lattice vertex algebra V_Q.

- $Y(\sigma v, z) = Y(v, e^{2\pi \mathrm{i}} z)$ for $v \in V_Q$.
- $\mathcal{D}_{X_N} \in$ (untwisted) representation of $\mathcal{W}_{X_N}$.

Theorem 2

$\mathcal{D}_{X_N}$ is a vacuum vector for the representation of $\mathcal{W}_{X_N}$, i.e., $Y(v, z)\mathcal{D}_{X_N}$ is regular in z for all $v \in \mathcal{W}_{X_N}$.

- **Virasoro constraints:** $Y(\omega, z)\mathcal{D}_{X_N} = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\mathcal{D}_{X_N}$ is regular $\Leftrightarrow$ $L_n\mathcal{D}_{X_N} = 0$ for $n \geq -1$.

Givental's Formula

- $\mathcal{D}_{A_1} \equiv \mathcal{D}_{\mathrm{pt}} = \mathcal{D}_{\mathrm{pt}}(\varepsilon, Q)$ is a formal power series in $\varepsilon^{\pm 1}$ and $Q = (Q_0, Q_1, Q_2, \dots)$, $t_{2k+1} = Q_k/(2k+1)!!$.
- $\mathcal{D}_{\mathrm{pt}}^{\otimes N}$ depends on ε and $Q^i = (Q_0^i, Q_1^i, Q_2^i, \dots)$, $i = 1, \dots, N$.
- **Givental's formula:** $\mathcal{D}_{X_N}(\varepsilon, q) = C_t \hat{S}_t^{-1} \hat{\mathcal{R}}_t \, \mathcal{D}_{\mathrm{pt}}^{\otimes N}$.
  - $t \in \mathcal{T}$ – generic, $q = (q_0, q_1, q_2, \dots)$, $q_k = (q_k^1, \dots, q_k^N)$.
  - $\mathcal{D}_{X_N} \in \mathbb{C}((\varepsilon))[[q_0 - t, q_1 + 1, q_2, \dots]]$.
  - ε, Q^i in $\mathcal{D}_{\mathrm{pt}}^{\otimes N}$ need to be rescaled (by $\sqrt{\Delta_i}$).
  - $\hat{S}_t$ and $\hat{\mathcal{R}}_t = \hat{\Psi}_t \hat{R}_t e^{\widehat{U_t/z}}$ are certain operators.
  - $\hat{\Psi}_t$ is a change of variables $(Q^1, \dots, Q^N) \mapsto q$.
  - $C_t = \mathrm{const}$ so that $\mathcal{D}_{X_N}$ is independent of t (omitted from now on).

Computation

- Ancestor potential $\mathcal{A}_t = \hat{\mathcal{R}}_t \, \mathcal{D}_{\mathrm{pt}}^{\otimes N} \Rightarrow \mathcal{D}_{X_N} = \hat{S}_t^{-1} \mathcal{A}_t$.
- $\hat{S}_t Y(v, \lambda) \mathcal{D}_{X_N} = \hat{S}_t Y(v, \lambda) \hat{S}_t^{-1} \mathcal{A}_t = Y_t(v, \lambda) \mathcal{A}_t$.
- Regular at $\lambda = \infty \Leftrightarrow$ regular at the critical values $\lambda = u_i(t)$ $(i = 1, \ldots, N)$ of $f_t(x)$ for a generic $t \in \mathcal{T}$.
- **Givental:** $Y_t(e^\alpha, \lambda) = \hat{\mathcal{R}}_t Y_{A_1}(e^\alpha, \lambda)_i \, \hat{\mathcal{R}}_t^{-1}$ for λ near $u_i(t)$.
  - $\alpha \in R$ is a cycle vanishing over $(t, u_i(t)) \in \Sigma$.
  - $Y_{A_1}(e^\alpha, \lambda)$ corresponds to the A_1 root lattice $\mathbb{Z}\alpha$.
  - $Y_{A_1}(e^\alpha, \lambda)_i$ denotes action on the i-th factor in $\mathcal{D}_{\mathrm{pt}}^{\otimes N}$.
- $Y(e^\alpha, \lambda) \mathcal{D}_{X_N} = \hat{S}_t^{-1} Y_t(e^\alpha, \lambda) \hat{\mathcal{R}}_t \mathcal{D}_{\mathrm{pt}}^{\otimes N} = \hat{S}_t^{-1} \hat{\mathcal{R}}_t Y_{A_1}(e^\alpha, \lambda)_i \, \mathcal{D}_{\mathrm{pt}}^{\otimes N}$.

Computation

- $Y(v,\lambda)=\sum_k Y(v_\perp^k,\lambda)Y(v_\alpha^k,\lambda), \quad v\in\mathcal{W}_{X_N}.$
  - $v_\perp^k\in$ Fock space of $\alpha^\perp\subset\mathfrak{h}$.
  - $v_\alpha^k\in$ Fock space of $\mathbb{C}\alpha$.
  - $v_\alpha^k\in \operatorname{Ker}\operatorname{Res}_{z=0} Y_{A_1}(e^{\pm\alpha},\lambda)=\mathcal{W}_{A_1}=$ Virasoro vertex algebra.
- $Y(v_\perp^k,\lambda)$ regular at $\lambda=u_i(t)$ since $r_\alpha v_\perp^k=v_\perp^k$ (no monodromy).
- $Y(v_\alpha^k,\lambda)\mathcal{D}_{X_N}=\hat{S}_t^{-1}\hat{\mathcal{R}}_t Y_{A_1}(v_\alpha^k,\lambda)_i\,\mathcal{D}_{\mathrm{pt}}^{\otimes N}$
  - Regular at $\lambda=u_i(t)$ because $\mathcal{D}_{\mathrm{pt}}$ satisfies Virasoro constraints.
- $Y(v,\lambda)\mathcal{D}_{X_N}$ is regular at $\lambda=u_i(t)$ for $v\in\mathcal{W}_{X_N}$.