Reflection Groups, Root Systems, and Singularities

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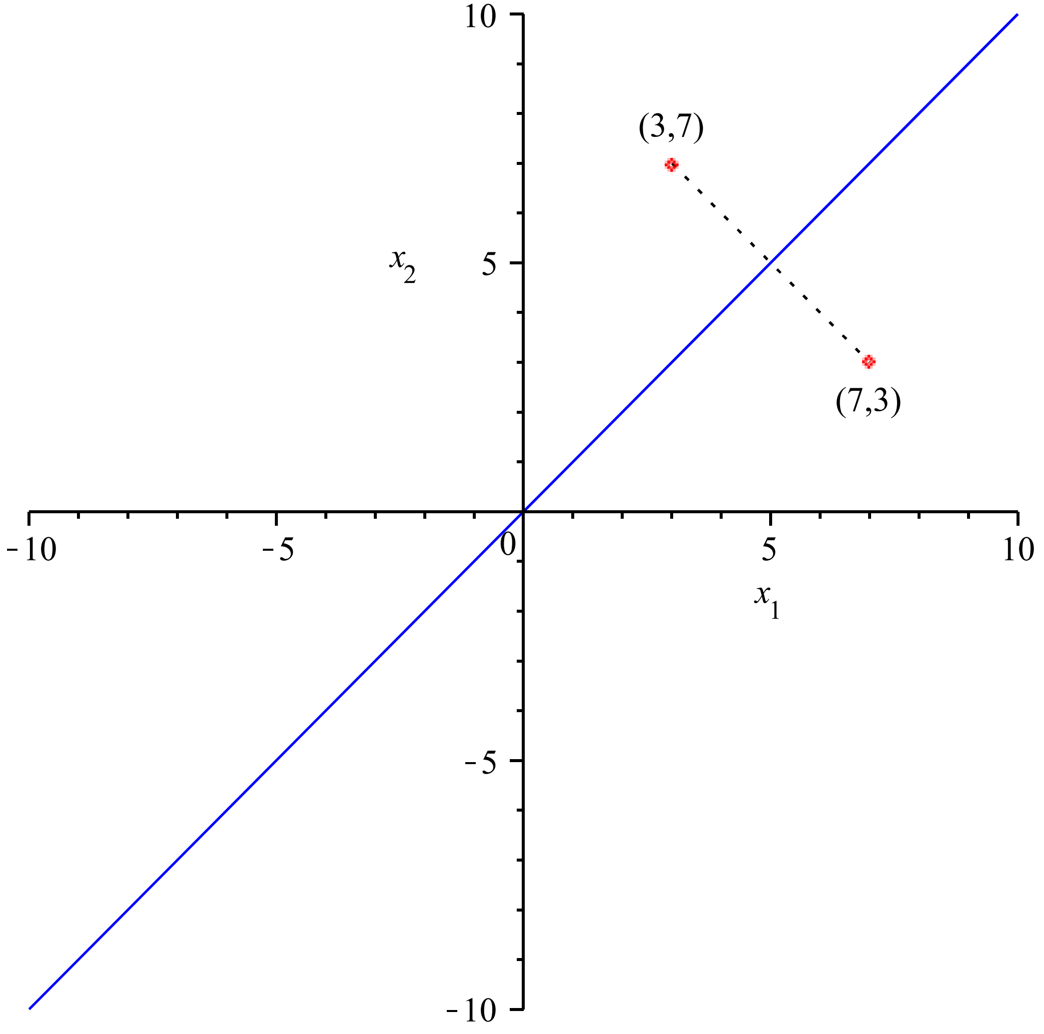
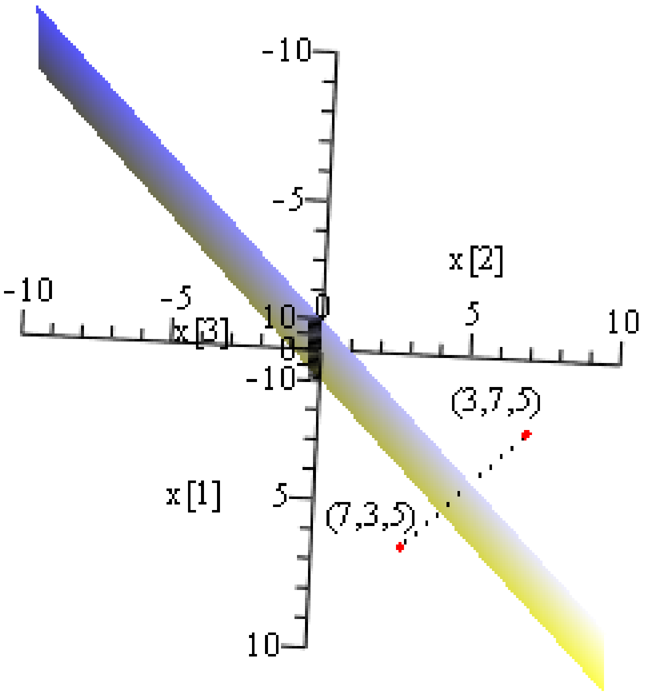
Reflection Groups

Reflection with respect to the line $x_1 = x_2$ flips the coordinates $(x_1, x_2) \to (x_2, x_1)$



Reflection with respect to the plane $x_1 = x_2$ flips $(x_1, x_2, x_3) \rightarrow (x_2, x_1, x_3)$



Similarly, reflection with respect to the plane $x_1 = x_3$ flips $(x_1, x_2, x_3) \to (x_3, x_2, x_1)$ and reflection with respect to the plane $x_2 = x_3$ flips $(x_1, x_2, x_3) \to (x_1, x_3, x_2)$.

Taking compositions we obtain all 6 permutations of the coordinates x_1, x_2, x_3 :

identity: $(x_1, x_2, x_3) \to (x_1, x_2, x_3)$,

flips: $(x_1, x_2, x_3) \to (x_2, x_1, x_3)$, $(x_1, x_2, x_3) \to (x_3, x_2, x_1)$, $(x_1, x_2, x_3) \to (x_1, x_3, x_2)$,

rotations: $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$, $(x_1, x_2, x_3) \to (x_3, x_1, x_2)$.

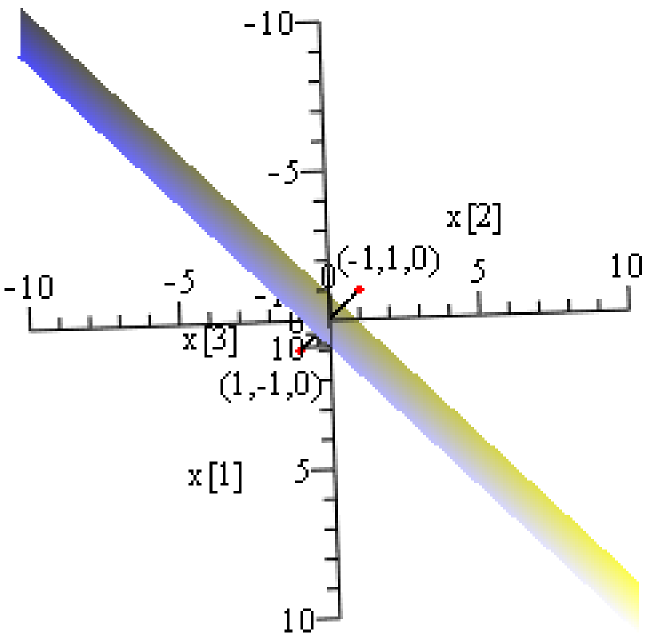
They form the symmetric group S_3.

Group: the composition of two permutations is a permutation, and the inverse of a permutation is a permutation.

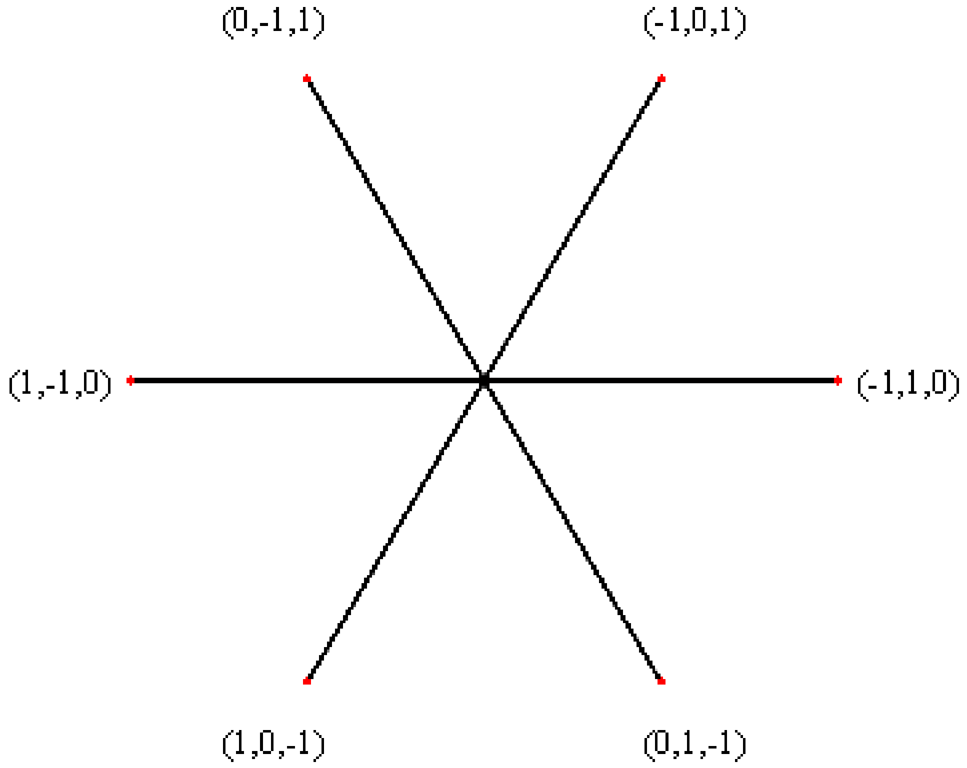
The symmetric group S_n has $n! = 1 \cdot 2 \cdot 3 \cdots n$ elements.

Root Systems

A plane is determined by the two vectors of length $\sqrt{2}$ perpendicular to it.



For the 3 planes $x_1 = x_2$, $x_1 = x_3$, $x_2 = x_3$ we obtain 6 vectors:



This is a root system of type A_2 consisting of the vectors

$$\alpha_1 = (1,-1,0),\quad -\alpha_1 = (-1,1,0),\quad \alpha_2 = (0,1,-1),\quad -\alpha_2 = (0,-1,1),\quad \alpha_1 + \alpha_2 = (1,0,-1),\quad -\alpha_1 - \alpha_2 = (-1,0,1)$$

The A_3 root system consists of 12 vectors:

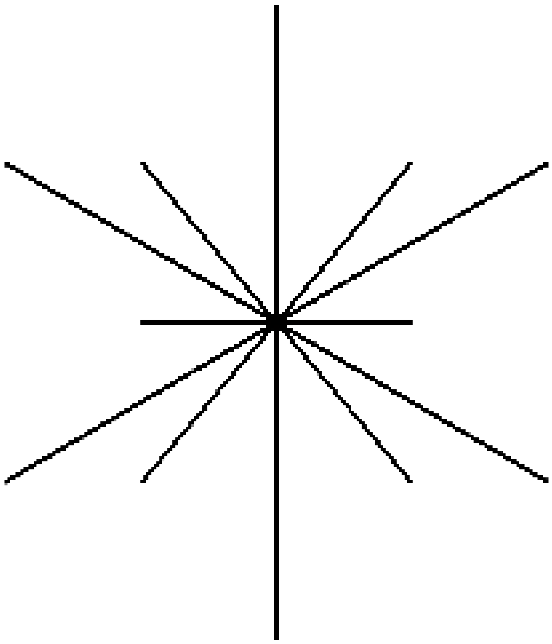
$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3,$ and their inverses.

The image of a root vector under one of our reflections is again a root vector.

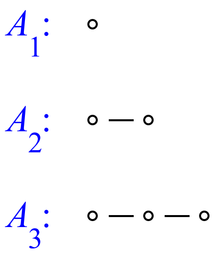
The length of each α_i is $\sqrt{2}$. The angles between α_1, α_2 and α_2, α_3 are 120°.

The vectors α_1 and α_3 are orthogonal. This means that we have dot products

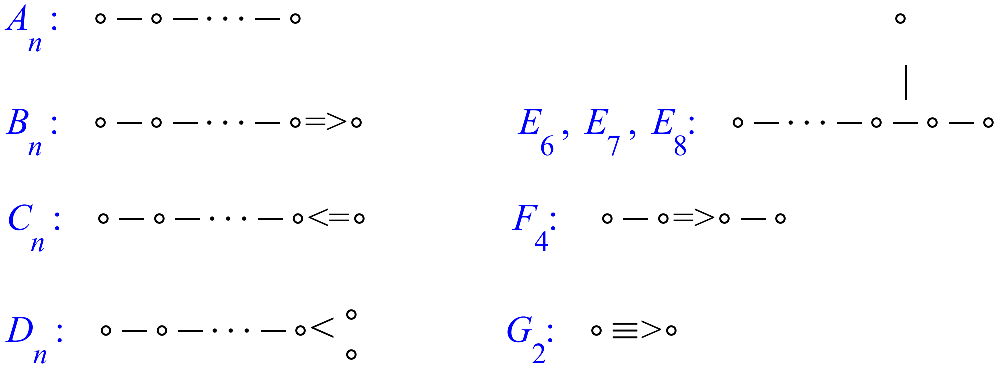
$$\alpha_1 \cdot \alpha_1 = \alpha_2 \cdot \alpha_2 = \alpha_3 \cdot \alpha_3 = 2, \qquad \alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = -1, \qquad \alpha_1 \cdot \alpha_3 = 0.$$



To the root systems we assign Dynkin diagrams



All root systems and Dynkin diagrams are:



They correspond to simple Lie algebras.

▼ Equations

The quadratic equation $x^2 + b\,x + c = 0$ has two solutions:

$$x_1 = -\frac{1}{2}\,b + \frac{1}{2}\sqrt{b^2 - 4\,c}\,, \qquad x_2 = -\frac{1}{2}\,b - \frac{1}{2}\sqrt{b^2 - 4\,c}$$

provided the discriminant $b^2 - 4\,c \geq 0$. Note that:

$$x^2 + b\,x + c = (x - x_1)\,(x - x_2) = x^2 + (-x_1 - x_2)\,x + x_1\,x_2$$

and hence

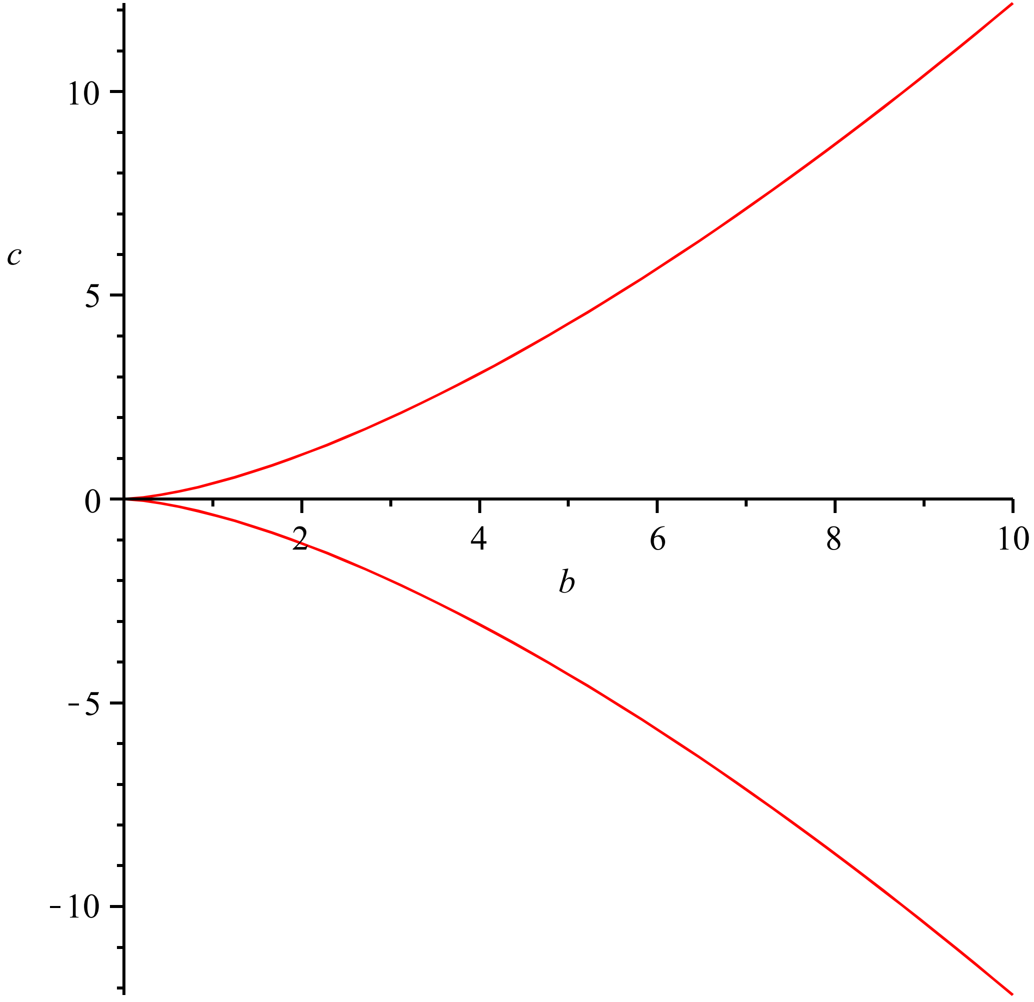
$$b = -x_1 - x_2, \qquad c = x_1\,x_2, \qquad b^2 - 4\,c = (x_1 - x_2)^2.$$

The cubic equation $x^3 - b\,x + c = 0$ has three distinct real solutions x_1, x_2, x_3

precisely when the discriminant $4\,b^3 - 27\,c^2 > 0$. It has a multiple root when the discriminant is 0.
We have:

$$x_1 + x_2 + x_3 = 0\,, \qquad x_1\,x_2 + x_1\,x_3 + x_2\,x_3 = -b\,, \qquad x_1\,x_2\,x_3 = -c\,,$$

$$4\,b^3 - 27\,c^2 = (x_1 - x_2)^2\,(x_1 - x_3)^2\,(x_2 - x_3)^2$$

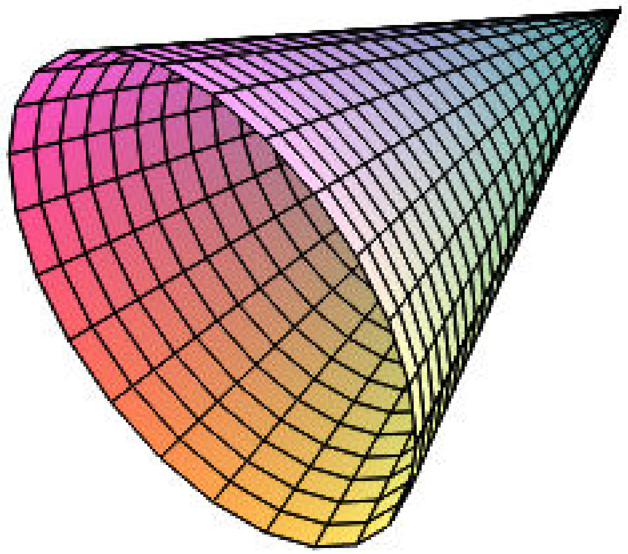


The discriminant of a cubic equation

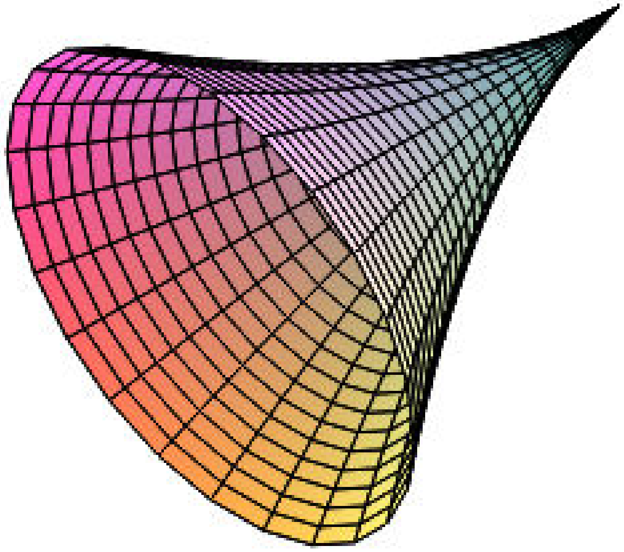
Singularities of Surfaces

The surface given by the equation $x^{n+1} = y^2 + z^2$ has a singularity at the origin $(0, 0, 0)$ called an A_n-singularity. There are singularities corresponding to all Dynkin diagrams.

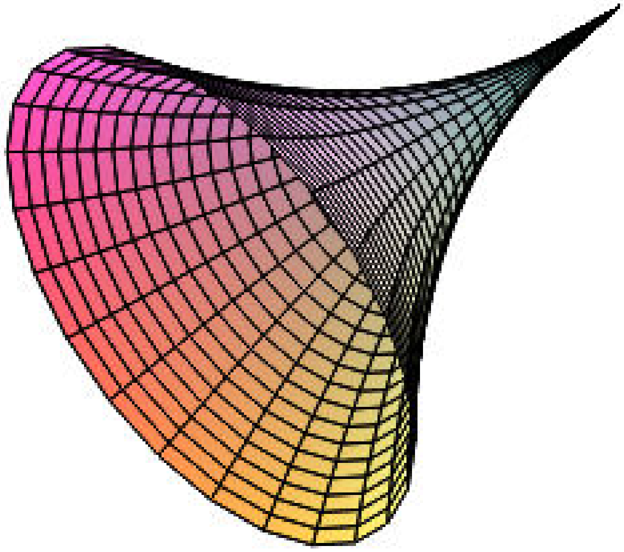
A1-singularity



A2-singularity



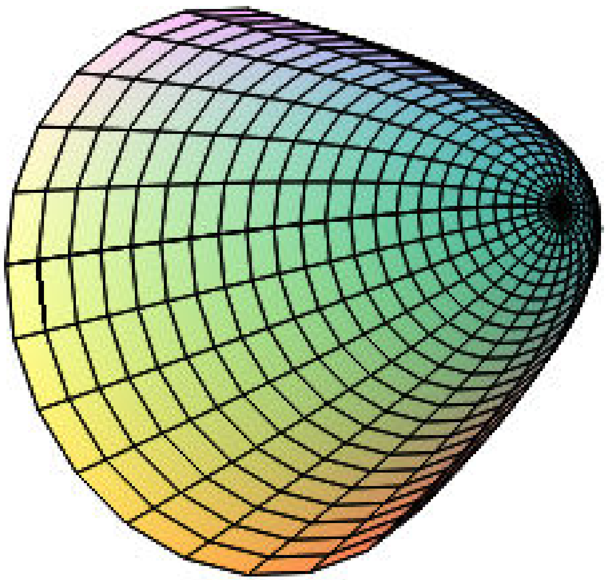
A3-singularity



Deforming Singularities

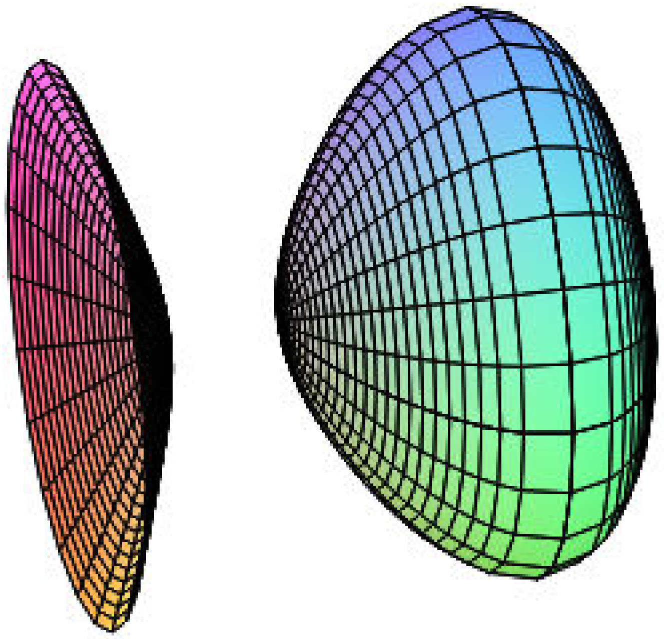
After deforming the A_1 singularity $x^2 = y^2 + z^2$ to the surface $x^2 + c = y^2 + z^2$ it becomes non-singular (smooth) for $c < 0$ when the discriminant > 0.

Deformed A1-singularity with discriminant >0

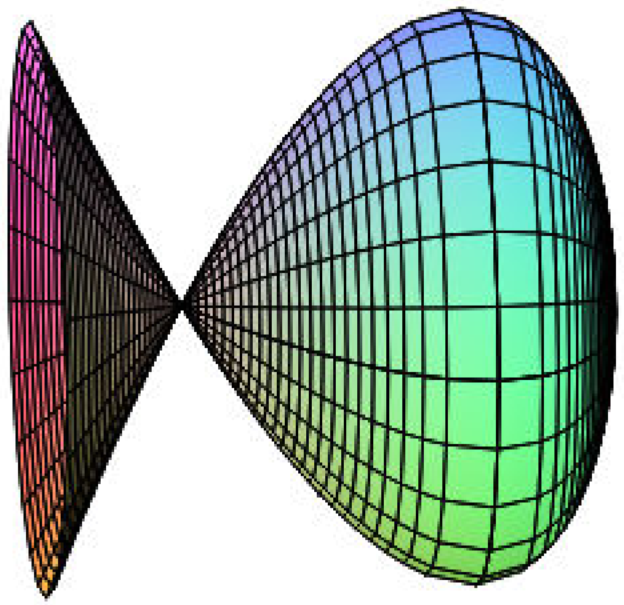


We deform the A_2 singularity $x^3=y^2+z^2$ to the surface $x^3-bx+c=y^2+z^2$. Then it is non-singular when the discriminant >0 and singular when the discriminant $=0$.

Deformed A2-singularity with discriminant >0



Deformed A2-singularity with discriminant =0



Here is a deformed the A_3 singularity with discriminant $=0$. Notice the resemblance with the Dynkin diagrams $\circ$, $\circ - \circ$, $\circ - \circ - \circ$. This holds in general for types ADE.

Deformed A3-singularity with discriminant =0