

Singularities, Root Systems, and \mathcal{W} -Algebras

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Gromov–Witten Theory

- **Gromov–Witten invariants** of a manifold $X \Leftrightarrow$ generating series \mathcal{D}_X – total descendant potential.
- **Witten–Kontsevich** function $\mathcal{D}_{\text{pt}} \Leftrightarrow$ intersection theory on $\overline{\mathcal{M}}_{g,n}$.
 - ▶ τ -function for the KdV hierarchy.
 - ▶ Virasoro constraints $L_n \mathcal{D}_{\text{pt}} = 0$ ($n \geq -1$). String eq. $L_{-1} \mathcal{D}_{\text{pt}} = 0$.
 - ▶ The KdV hierarchy + string eq. determine uniquely \mathcal{D}_{pt} .
- **Genus 0** Gromov–Witten invariants of $X \Rightarrow$ quantum cup product on $H^*(X) \Rightarrow$ Frobenius algebra + flat connection \Rightarrow Frobenius manifold.
- **Givental’s formula** expresses \mathcal{D}_X in terms of \mathcal{D}_{pt} and the semisimple Frobenius structure.
 - ▶ Virasoro constraints for $\mathcal{D}_{\text{pt}} \Rightarrow$ Virasoro constraints for \mathcal{D}_X .

Main Questions

Questions

- Does \mathcal{D}_X satisfy an integrable hierarchy?
- Does \mathcal{D}_X satisfy Virasoro or \mathcal{W} -constraints?

Generalized Witten Conjecture:

- The total descendant potential for h -spin curves is a τ -function for the h -th Gelfand–Dickey hierarchy and satisfies \mathcal{W}_h -constraints.
 - ▶ $\mathcal{W}_h = \mathcal{W}(2, 3, \dots, h)$ is the Zamolodchikov–Fateev–Lukyanov \mathcal{W} -algebra.
 - ▶ $\mathcal{W}_2 =$ Virasoro algebra.

Previous Work

- **Adler–van Moerbeke:** There is a unique τ -function for the h -th Gelfand–Dickey hierarchy solving the string equation. It satisfies the \mathcal{W}_h -constraints.
- **K. Saito:** Singularity theory \Rightarrow semisimple Frobenius structure \Rightarrow total descendant potential \mathcal{D}_{X_N} defined by Givental's formula.
 - ▶ X_N is the type of the singularity.
 - ▶ $X_N = A_N, D_N, E_6, E_7$ or E_8 for simple singularities.
- **Fan–Jarvis–Ruan:** \mathcal{D}_{A_N} is the total descendant potential for Gromov–Witten invariants of h -spin curves ($h = N + 1$).
 - ▶ Generalization for types D and E .
- **E. Frenkel–Givental–Milanov:** \mathcal{D}_{X_N} ($X = A, D, E$) is a τ -function for the **Kac–Wakimoto** hierarchy of type X_N in the principal realization.
 - ▶ Type $A_N \Leftrightarrow h$ -th Gelfand–Dickey hierarchy ($h = N + 1$).
 - ▶ This implies the generalized Witten conjecture.

Main Result

Theorem

The total descendant potential \mathcal{D}_{X_N} ($X = A, D, E$) of a simple singularity satisfies \mathcal{W}_{X_N} -constraints.

Remarks

- We can prove both the integrable hierarchy and \mathcal{W} -constraints.
- The proof works more generally for weighted homogeneous singularities.
- The proof uses singularity theory, Givental's formula, and vertex algebras.

Simple Singularities

- $f(x_1, x_2, x_3)$ – homogeneous polynomial of degree h ($\deg x_i = a_i$) with an isolated **critical point** at $x = 0$ ($\frac{\partial f}{\partial x_i}(0) = 0$).

Type	h	a_1, a_2, a_3	$f(\mathbf{x})$	Exponents
A_N	$N+1$	$1, a, N+1-a$	$x_1^{N+1} + x_2 x_3$	$1, 2, \dots, N$
D_N	$2N-2$	$2, N-2, N-1$	$x_1^{N-1} + x_1 x_2^2 + x_3^2$	$1, 3, \dots, 2N-3, N-1$
E_6	12	3, 4, 6	$x_1^4 + x_2^3 + x_3^2$	1, 4, 5, 7, 8, 11
E_7	18	4, 6, 9	$x_1^3 x_2 + x_2^3 + x_3^2$	1, 5, 7, 9, 11, 13, 17
E_8	30	6, 10, 15	$x_1^5 + x_2^3 + x_3^2$	1, 7, 11, 13, 17, 19, 23, 29

Milnor Fibration

- **Miniversal deformations:** $f_t(x) = f(x) + \sum_{i=1}^N t_i g_i(x)$.
 - ▶ $t = (t_1, \dots, t_N) \in \mathcal{T} = \mathbb{C}^N$.
 - ▶ $g_i(x)$ – homogeneous polynomials giving a basis of $\mathbb{C}[x]/(\partial f(x))$.
- **Milnor fibration:** $\mathcal{T} \times \mathbb{C}^3 \rightarrow \mathcal{T} \times \mathbb{C}$, $(t, x) \mapsto (t, f_t(x))$.
 - ▶ Fibers X_s ($s = (t, \lambda)$) consist of $(t, x) \in \mathcal{T} \times \mathbb{C}^3$ such that $f_t(x) = \lambda$.
 - ▶ Smooth fibration outside the **discriminant** Σ consisting of s such that X_s is singular.
 - ▶ All smooth fibers are diffeomorphic to X_1 , $1 = (0, 1) \in \mathcal{T} \times \mathbb{C}$.
- **Milnor lattice:** $Q = H_2(X_1; \mathbb{Z})$ with negative the intersection form \cong root lattice of type X_N .
 - ▶ Homomorphism $Q \rightarrow H_2(X_s; \mathbb{Z})$ for generic $s \in \Sigma$ and every path from 1 to s avoiding Σ .
 - ▶ $\alpha \in Q$ is a **vanishing cycle** if $\alpha \mapsto 0$ and $(\alpha|\alpha) = 2$.
 - ▶ The set R of vanishing cycles is a root system of type X_N .

Monodromy Representation

- **Fundamental group** of $(\mathcal{T} \times \mathbb{C}) \setminus \Sigma$ acts on Q by orthogonal transformations. Image = Weyl group W of type X_N .
- **Picard–Lefschetz:** Small loop around $s \in \Sigma \Rightarrow$ reflection $r_\alpha \in W$ ($\alpha \in R$ vanishing over s).
 - ▶ $r_\alpha(\beta) = \beta - (\alpha|\beta)\alpha$.
 - ▶ W is generated by r_α ($\alpha \in R$).
- **Classical monodromy:** Big loop around $\Sigma \Rightarrow$ **Coxeter element** $\sigma \in W$.
 - ▶ $\sigma = r_{\alpha_1} \cdots r_{\alpha_N}$ where $\alpha_1, \dots, \alpha_N$ is a basis of simple roots.
 - ▶ $|\sigma| = h$ is the Coxeter number.
 - ▶ σ is diagonalizable on $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ with eigenvalues $e^{2\pi i m_k/h}$.
 - ▶ σ has no fixed points in \mathfrak{h} .

Affine Lie Algebras

- \mathfrak{g} – finite-dimensional Lie algebra with a symmetric invariant bilinear form $(\cdot|\cdot)$.
 - ▶ E.g., Killing form, or $(a|b) = \text{tr}(ab)$ for $\mathfrak{g} \subseteq \mathfrak{gl}_n$.
- **Loop algebra** $\mathfrak{g}[t, t^{-1}] =$ polynomial maps from S^1 to \mathfrak{g} .
 - ▶ $[a(t), b(t)] = [a, b](t)$,
 - ▶ $[a_m, b_n] = [a, b]_{m+n}$, where $a_m = at^m$.
- **Affine Lie algebra** $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ is a central extension of the loop algebra:
 - ▶ K is central,
 - ▶ $[a_m, b_n] = [a, b]_{m+n} + m\delta_{m,-n}(a|b)K$,
 - ▶ $\mathfrak{g} \subset \hat{\mathfrak{g}}$ via $a \mapsto a_0$.
- **Basic representation** $V(\Lambda_0)$ – irreducible representation of $\hat{\mathfrak{g}}$:
 - ▶ $V(\Lambda_0)$ is generated by a vector $\mathbf{1}$ such that $\mathfrak{g}[t]\mathbf{1} = \{0\}$,
 - ▶ K acts as the identity on $V(\Lambda_0)$,
 - ▶ $V(\Lambda_0)$ has the structure of a vertex algebra.

Lattice Vertex Algebras

Definition (Borcherds)

A **vertex algebra** is a vector space V (space of states) with a vacuum vector $\mathbf{1} \in V$ and state-field correspondence $Y(\cdot, z) \cdot : V \otimes V \rightarrow V((z))$ + axioms.

Notation

- Q – integral (even) lattice.
- $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ – vector space with symmetric bilinear form $(\cdot | \cdot)$.
- $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ – Heisenberg Lie algebra.
- $B = V(\Lambda_0) \cong \mathcal{S}(\mathfrak{h}[t^{-1}]t^{-1})$ – bosonic Fock space.
- $B_\alpha \cong B$ as an $\hat{\mathfrak{h}}$ module except that the \mathfrak{h} -action has weight $\alpha \in Q$.

Definition (Borcherds)

Lattice vertex algebra $V_Q = \bigoplus_{\alpha \in Q} B_\alpha$.

The Vertex Algebra \mathcal{W}_{X_N}

- **I. Frenkel–Kac:** When Q is a root lattice of type X_N ($X = A, D, E$), $V_Q \cong V(\Lambda_0)$ – basic representation of $\hat{\mathfrak{g}}$.
 - ▶ \mathfrak{g} acts by derivations on V_Q via $a \mapsto a_0$.
- **E. Frenkel–Kac–Radul–Wang:** The \mathcal{W} -algebra \mathcal{W}_{X_N} (with central charge N) \cong \mathfrak{g} -invariant part of $V(\Lambda_0)$.
 - ▶ $\mathcal{W}_{X_N} \subset B$ is fixed pointwise by the Weyl group W .
 - ▶ \mathcal{W}_{X_N} is freely generated by fields of conformal weights $m_1 + 1, \dots, m_N + 1$ (**Feigin–E. Frenkel**).
 - ▶ \mathcal{W}_{X_N} contains a Virasoro field; $m_1 + 1 = 2$.
 - ▶ $\mathcal{W}_{A_N} = \mathcal{W}_{N+1}$ (central charge N); freely generated by fields of conformal weights $2, \dots, N + 1$.
 - ▶ $\mathcal{W}_{A_1} =$ Virasoro vertex algebra (central charge 1).

Main Results

Theorem 1

The total descendant potential \mathcal{D}_{X_N} ($X = A, D, E$) lies in a σ -twisted representation of the lattice vertex algebra V_Q .

- $Y(\sigma v, z) = Y(v, e^{2\pi i} z)$ for $v \in V_Q$.
- $\mathcal{D}_{X_N} \in$ (untwisted) representation of \mathcal{W}_{X_N} .

Theorem 2

\mathcal{D}_{X_N} is a *vacuum vector* for the representation of \mathcal{W}_{X_N} , i.e., $Y(v, z)\mathcal{D}_{X_N}$ is regular in z for all $v \in \mathcal{W}_{X_N}$.

- **Virasoro constraints:** $Y(\omega, z)\mathcal{D}_{X_N} = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \mathcal{D}_{X_N}$ is regular $\Leftrightarrow L_n \mathcal{D}_{X_N} = 0$ for $n \geq -1$.