

Due Friday at 2:25 pm, unless stated otherwise.

1.(*due 1/16*) The Lie algebra $\mathfrak{gl}(n)$ has a basis $\{E_{ij}\}$ where E_{ij} is the n by n matrix with an (i, j) -entry one and all other entries zero. Check that the map $E_{ij} \mapsto x_i \partial_{x_j}$ defines a representation of $\mathfrak{gl}(n)$ in the space $\mathbb{C}[x_1, \dots, x_n]$.

2.(*due 1/16*) Prove that the set of $n \times n$ orthogonal (respectively unitary) matrices is a subgroup of $GL(n, \mathbb{R})$ (respectively $GL(n, \mathbb{C})$).

3.(*due 1/23*) Consider a complex Lie algebra \mathfrak{g} with an anti-linear anti-involution ω . Show that the set of all $A \in \mathfrak{g}$ such that $\omega(A) = -A$ is a real Lie subalgebra of \mathfrak{g} .

4.(*due 1/23*) Prove that any finite-dimensional unitary representation is completely reducible, i.e., is a direct sum of irreducible ones.

5.(*due 1/23*) Prove that on a finite-dimensional vector space there are no linear operators A, B with $[A, B] = I$ (identity operator).

6.(*due 1/30*) Show that the graded dimension of the bosonic Fock space is

$$\sum_{m=0}^{\infty} (\dim B_m) q^m = \frac{1}{\varphi(q)}, \quad \text{where } \varphi(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

7.(*due 1/30*) Prove that $[d_m, a_n] = -na_{m+n}$ where a_n are the generators of the Heisenberg algebra and $d_m = (1/2) \sum_{k \in \mathbb{Z}} :a_{-k} a_{m+k}:$.

8.(*due 1/30*) Using Problem 7, show that $[d_m, d_{-m}] = 2md_0 + (m^3 - m)/12$.

9.(*due 2/6*) Show that $\omega(L_n) = L_{-n}$, $\omega(C) = C$ defines an anti-involution of the Virasoro algebra.

10.(*due 2/6*) Show that the Fock space representation of the Virasoro algebra is unitary.

11.(*due 2/13*) Show that for $\mu = 0$, the highest component $B' = V(1, 0)$ of the Fock space is a proper submodule.

12.(*due 2/13*) Find all singular vectors in $M(c, h)_{h+3}$.

13. (due 2/20) Compute $\det_3(c, h)$ directly from the definition.

14. (due 2/20) Compute $\det_3(c, h)$ using the Kac determinant formula.

15. (due 2/27) Let \mathfrak{g} be a Lie algebra, and V and W be representations of \mathfrak{g} . Define an action of \mathfrak{g} on the tensor product $V \otimes W$ by the formula $a(v \otimes w) = av \otimes w + v \otimes aw$. Check that $V \otimes W$ is a representation of \mathfrak{g} .

16. (due 2/27) Let \mathfrak{g} be a Lie algebra equipped with an anti-linear anti-involution ω , and V and W be unitary representations of \mathfrak{g} . Show that the representation $V \otimes W$ is unitary for the Hermitian form $(v \otimes w, v' \otimes w') = (v, v')(w, w')$.

17. (due 3/20) Consider a Lie algebra \mathfrak{g} with a bilinear map $\alpha: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$, and define a Lie bracket on $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}K$ by letting K be central and $[a, b]_{\hat{\mathfrak{g}}} = [a, b]_{\mathfrak{g}} + \alpha(a, b)K$ for $a, b \in \mathfrak{g}$. Derive necessary and sufficient conditions for α so that $\hat{\mathfrak{g}}$ is a Lie algebra.

18. (due 3/20) Fix $\alpha, \beta \in \mathbb{C}$. Check that the operators d_n ($n \in \mathbb{Z}$) defined on $V = \text{span}\{v_k\}_{k \in \mathbb{Z}}$ by $d_n v_k = (k - \alpha - \beta(n + 1))v_{k-n}$ satisfy $[d_m, d_n] = (m - n)d_{m+n}$.

19. (due 3/20) With the notation from Problem 18, verify that $L_n = d_n + \delta_{n,0}\alpha(\alpha + 2\beta - 1)/2$ provide an embedding of the Virasoro algebra with central charge $c = -12\beta^2 + 12\beta - 2$ into $\widehat{\mathfrak{gl}}_{\infty}$.

20. (due 3/27) Let ψ_n^+ be the operator of wedging with v_{-n} , and ψ_n^- be the operator of contracting v_{n+1} , acting on the semi-infinite wedge space. Compute the anticommutator of ψ_m^+ and ψ_n^- ($m, n \in \mathbb{Z}$).

21. (due 3/27) Consider the elements $d_n \in \widetilde{\mathfrak{gl}}_{\infty}$ defined in Problem 18 for $\alpha = 0$, and denote by $L_n^{\beta} = \hat{r}(d_n)$ the corresponding linear operators on the semi-infinite wedge space. Introduce the generating series

$$\psi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \psi_n^{\pm} z^{-n-1}, \quad L^{\beta}(z) = \sum_{n \in \mathbb{Z}} L_n^{\beta} z^{-n-2}.$$

Prove that

$$L^{\beta}(z) = \beta :(\partial_z \psi^+(z)) \psi^-(z): + (\beta - 1) : \psi^+(z) \partial_z \psi^-(z) :.$$

22. (due 4/3) With the notation from Problem 21, compute $[L_0^{\beta}, \psi_n^+]$ and $[L_0^{\beta}, \psi_n^-]$.

23. (due 4/3) Derive from the Jacobi triple product identity

$$\prod_{n=1}^{\infty} (1 - q^n)(1 - zq^{n-1})(1 - z^{-1}q^n) = \sum_{m \in \mathbb{Z}} (-z)^m q^{m(m-1)/2}$$

the Euler identity

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{m \in \mathbb{Z}} (-1)^m q^{m(3m+1)/2}$$

and the Gauss identity

$$\prod_{n=1}^{\infty} \frac{1 - q^n}{1 + q^n} = \sum_{m \in \mathbb{Z}} (-q)^{m^2}.$$

24. (due 4/17) Prove the following formula for the representation of GL_{∞} in $F^{(0)}$:

$$R(A)(v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \cdots) = \sum (\det A_{j_0, j_{-1}, \dots}^{i_0, i_{-1}, \dots}) v_{j_0} \wedge v_{j_{-1}} \wedge v_{j_{-2}} \wedge \cdots,$$

where the sum is over all integers $j_0 > j_{-1} > \cdots$ such that $j_{-t} = -t$ for $t \gg 0$, and $A_{j_0, j_{-1}, \dots}^{i_0, i_{-1}, \dots}$ denotes the submatrix of A obtained by intersecting rows j_0, j_{-1}, \dots and columns i_0, i_{-1}, \dots .

25. (due 4/17) Prove that the linear operator

$$\Omega = \sum_{n \in \mathbb{Z}} \psi_{-n}^+ \otimes \psi_{n-1}^- = \text{Res}_z \psi^+(z) \otimes \psi^-(z)$$

commutes with the action of \mathfrak{gl}_{∞} on $F \otimes F$.

26. (due 4/17) Derive an explicit formula for the Hirota bilinear operator

$$x^n (f \bullet g) := \partial_u^n (f(x-u)g(x+u)) \Big|_{u=0}.$$

(Here x and u are single variables, not vectors.)