Due Thursday at 11:45 am, unless stated otherwise.

1. (due 1/17) Rewrite the definition of vertex algebra in terms of modes.
   Hint: The field condition can be stated as $a(n)b = 0$ for $n \gg 0$ ($n \geq 0$ large enough, depending on $a$ and $b$).

2. (due 1/17) Prove that in any vertex algebra one has: $Y(a, z)1 = e^{zT}a$. Equivalently, $a_{(-n-1)}1 = T^{(n)}a$ for $n \geq 0$, where $T^{(n)} := T^n/n!$.

3. (due 1/17) Let $V$ be a commutative vertex algebra. Prove that the product $ab := a_{(-1)}b$ makes $V$ a commutative associative algebra with a unit $1 = 1$ and a derivation $T$.

4. (due 1/24) Let
   
   $$ a(z, w) = \sum_{m,n\in\mathbb{Z}} a_{(m,n)} z^{-m-1} w^{-n-1} $$

   and

   $$ c^j(w) = \sum_{k\in\mathbb{Z}} c^j_{(k)} w^{-k-1} $$

   be formal distributions. Prove that the following three equations are equivalent:

   $$ a(z, w) = \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z, w), $$

   $$ \text{Res}_z z^m a(z, w) = \sum_{j=0}^{N-1} \binom{m}{j} w^{m-j} c^j(w), \quad m \in \mathbb{Z}, $$

   $$ a_{(m,n)} = \sum_{j=0}^{N-1} \binom{m}{j} c^j_{(m+n-j)}, \quad m, n \in \mathbb{Z}. $$

5. (due 1/24) Consider the formal distribution $L(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$, where the elements $L_m$ satisfy the Virasoro algebra commutation relations

   $$ [L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} c $$

   for some fixed $c \in \mathbb{C}$. Derive a formula for the commutator $[L(z), L(w)]$ that involves $L(w)$, the formal delta-function, and their derivatives.
6. (due 1/31) For two $L$-valued formal distributions (where $L$ is a Lie algebra) $a(z), b(z) \in L[[z, z^{-1}]]$, their $\lambda$-bracket is defined as $[a_\lambda b](z) := \text{Res}_x e^{\lambda(x-z)}[a(x), b(z)]$. Prove that

$$[(\partial a)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda(\partial b)] = (\lambda + \partial a_\lambda b]$$

where $\partial = \partial_z$.

7. (due 1/31) Consider the $\mathbb{C}[\partial]$-module $R = \mathbb{C}[\partial]L + \mathbb{C}C$, where $\partial C = 0$, and define a $\lambda$-bracket on it by

$$[L_\lambda L] = (\partial + 2\lambda)L + \lambda^3C/12, \quad [C_\lambda C] = [C_\lambda L] = 0.$$

Verify that $R$ is a Lie conformal algebra.

8. (due 2/7) Prove that in a Lie conformal algebra $R$, every torsion element is central. In other words, if $p(\partial)c = 0$ for some $c \in R$ and a nonzero polynomial $p$, then $[c_\lambda a] = 0$ for all $a \in R$.

9. (due 2/7) Verify that $\mathfrak{gl}(m|n)$ satisfies the axioms of a Lie superalgebra. Let $q(n) \subset \mathfrak{gl}(n|n)$ be the set of matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. Check that $q(n)$ is a subalgebra of the Lie superalgebra $\mathfrak{gl}(n|n)$.

10. (due 2/14) Prove that the normally ordered product $:a(z)b(z):$ of two fields is a well-defined field.

11. (due 2/14) Consider formal distributions with values in an associative algebra.

(a) Prove that if $B(z, w)$ is local and it is local with respect to $c(x)$, then $\text{Res}_z B(z, w)$ is local with respect to $c(x)$.

(b) Prove that if the pairs $a(z), c(x)$ and $b(w), c(x)$ are local, then $a(z)b(w)$ is local with respect to $c(x)$.

12. (due 2/21) Prove that in any vertex algebra one has $Y(Ta, z) = \partial_z Y(a, z)$.

13. (due 2/21) Consider the Heisenberg Lie algebra $\text{Heis}$ with basis $\{a_n, K\}_{n \in \mathbb{Z}}$ and commutation relations $[a_m, a_n] = m\delta_{m,-n}K$. Let $B = \mathbb{C}[t_1, t_2, t_3, \ldots]$ be the bosonic Fock space.

(a) Prove that $a_{-n} \mapsto nt_n, \quad a_0 \mapsto 0, \quad a_n \mapsto \partial/\partial t_n, \quad K \mapsto 1 \quad (n > 0)$ gives an irreducible representation of $\text{Heis}$ on $B$. Note that the vector $\mathbf{1} := 1 \in B$ has the property $a_n \mathbf{1} = 0$ for $n \geq 0$.

(b) Let $V$ be an irreducible representation of $\text{Heis}$ with $K \mapsto 1$. Assume that there exists a nonzero vector $\mathbf{1} \in V$ such that $a_n \mathbf{1} = 0$ for $n \geq 0$. Prove that $V$ is isomorphic to the Fock space $B$ as a module over $\text{Heis}$. 
14. (due 3/14) Consider a free boson field $a(z)$, so that $[a(z), a(w)] = \partial_w \delta(z, w)$ and the field

$$L^\lambda(z) = \frac{1}{2} a(z)^2 + \left( \frac{1}{2} - \lambda \right) \partial_z a(z),$$

for a fixed $\lambda \in \mathbb{C}$. Prove that the modes of $L^\lambda(z)$ satisfy the commutation relations of the Virasoro algebra with central charge $c = -12\lambda^2 + 12\lambda - 2$.

15. (due 3/14) Let $V$ be a vertex algebra, in which there are two odd elements, $v_+$ and $v_-$, with the property that $\psi^\pm(z) = Y(v_\pm, z)$ is a pair of charged free fermions. More precisely, we assume the following (anti)commutators:

$$[\psi^+(z), \psi^+(w)] = [\psi^-(z), \psi^-(w)] = 0, \quad [\psi^+(z), \psi^-(w)] = \delta(z, w).$$

Let $\alpha(z) = :\psi^+(z)\psi^-(z): = Y(v_{+(-1)}v_-, z)$. Prove that:

$$[\alpha(z), \alpha(w)] = \partial_w \delta(z, w), \quad [\alpha(z), \psi^+[w)] = \pm \psi^+[w] \delta(z, w)$$

and

$$:\alpha(z)\alpha(z): = :\partial_z \psi^+(z)\psi^-(z): - :\psi^+(z)\partial_z \psi^-(z):.$$ 

16. (due 3/26) Consider the following fields in the lattice vertex algebra $V_{\sqrt{2}\mathbb{Z}}$:

$$h(z) = \sqrt{2} a(z) = Y(\sqrt{2}t_1, z), \quad e(z) = Y(|\sqrt{2}\rangle, z), \quad f(z) = Y(|-\sqrt{2}\rangle, z).$$

Derive formulas for the commutators of all these fields in terms of them and the delta-function.

17. (due 4/4) Denote by $V_{L,\varepsilon}$ the vertex algebra associated to an integral lattice $L$ and a 2-cocycle $\varepsilon$ on $L$. Prove that, if $\bar{\varepsilon}$ is a 2-cocycle equivalent to $\varepsilon$, then $V_{L,\varepsilon}$ is isomorphic to $V_{L,\bar{\varepsilon}}$.

18. (due 4/11) Let $\mathfrak{h}$ be a vector space with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$, and let $L \subset \mathfrak{h}$ be an integral lattice with rank $L = \dim \mathfrak{h} = r$. Choose a pair of dual bases $\{a^i\}, \{b^i\}$ of $\mathfrak{h}$, so that $(a^i|b^j) = \delta_{i,j}$. Recall that in the lattice vertex algebra $V_L$, one has the following Virasoro field with central charge $c = r$:

$$L(z) = \frac{1}{2} \sum_{i=1}^r :a^i(z)b^i(z):.$$ 

Prove that:

(a) All free bosons $h(z) (h \in \mathfrak{h})$ are primary fields of conformal weight 1.
(b) All $Y(|\alpha\rangle, z) (\alpha \in L)$ are primary fields of conformal weight $|\alpha|^2/2$. 