

Due Thursday at 10:15 am, unless stated otherwise.

1. (due 1/17) Rewrite the definition of vertex algebra in terms of modes.

Hint: The field condition can be stated as $a_{(n)}b = 0$ for $n \gg 0$ ($n \geq 0$ large enough, depending on a and b).

2. (due 1/17) Prove that in any vertex algebra one has: $Y(a, z)|0\rangle = e^{zT}a$. Equivalently, $a_{(-n-1)}|0\rangle = T^{(n)}a$ for $n \geq 0$, where $T^{(n)} := T^n/n!$.

3. (due 1/24) Let $a(z, w) = \sum_{m, n \in \mathbb{Z}} a_{(m, n)} z^{-m-1} w^{-n-1}$ and $c^j(w) = \sum_{k \in \mathbb{Z}} c_{(k)}^j w^{-k-1}$ be formal distributions. Prove that the following three equations are equivalent:

$$\begin{aligned} a(z, w) &= \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z-w), \\ \text{Res}_z z^m a(z, w) &= \sum_{j=0}^{N-1} \binom{m}{j} w^{m-j} c^j(w), \quad m \in \mathbb{Z}, \\ a_{(m, n)} &= \sum_{j=0}^{N-1} \binom{m}{j} c_{(m+n-j)}^j, \quad m, n \in \mathbb{Z}. \end{aligned}$$

4. (due 1/24) Consider the formal distribution $L(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$, where the elements L_m satisfy the Virasoro algebra commutation relations $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m, -n}(m^3 - m)C/12$. Derive a formula for the commutator $[L(z), L(w)]$ that involves the derivatives of the formal delta-function.

5. (due 1/31) For two \mathfrak{g} -valued formal distributions (where \mathfrak{g} is a Lie algebra) $a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]]$, their λ -bracket is defined as $[a_\lambda b](z) := \text{Res}_u e^{\lambda(u-z)}[a(u), b(z)]$. Prove that $[(\partial a)_\lambda b] = -\lambda[a_\lambda b]$ and $[a_\lambda(\partial b)] = (\lambda + \partial)[a_\lambda b]$, where $\partial = \partial_z$.

6. (due 1/31) Consider the $\mathbb{C}[\partial]$ -module $R = \mathbb{C}[\partial]\ell + \mathbb{C}c$, where $\partial c = 0$, and define a λ -bracket on it by $[\ell_\lambda \ell] := (\partial + 2\lambda)\ell + c\lambda^3/12$. Verify that R is a Lie conformal algebra.

7. (due 2/7) Consider the Heisenberg Lie algebra $\hat{\mathfrak{h}}$ with basis $\{a_n, K\}_{n \in \mathbb{Z}}$ and commutation relations $[a_m, a_n] = m\delta_{m, -n}K$. Let $F = \mathbb{C}[t_1, t_2, t_3, \dots]$ be the Fock space.

(a) Prove that $a_{-n} \mapsto nt_n$, $a_0 \mapsto 0$, $a_n \mapsto \partial/\partial t_n$, $K \mapsto 1$ ($n > 0$) gives an irreducible representation of $\hat{\mathfrak{h}}$ on F . Note that the vector $|0\rangle := 1 \in F$ has the property $a_n|0\rangle = 0$ for all $n \geq 0$.

(b) Let V be an irreducible representation of $\hat{\mathfrak{h}}$ with $K \mapsto 1$. Assume that there exists a nonzero vector $|0\rangle \in V$ such that $a_n|0\rangle = 0$ for all $n \geq 0$. Prove that V is isomorphic to the Fock space F as an $\hat{\mathfrak{h}}$ -module.

8. (due 2/7) Prove that the normally ordered product $:a(z)b(z):$ of two fields is a well-defined field.

9. (due 2/14) Prove that for all $n \in \mathbb{Z}$

$$a(w)_{(n)}b(w) = \text{Res}_z \left(a(z)b(w) i_{z,w}(z-w)^n - b(w)a(z) i_{w,z}(z-w)^n \right).$$

10. (due 2/21) Let $a(z)$ and $b(z)$ be two translation covariant fields on a vector space V equipped with $|0\rangle$ and T .

(a) Prove that the n -th product $a(z)_{(n)}b(z)$ is translation covariant for any $n \in \mathbb{Z}$.

(b) Prove that $(a(z)_{(n)}b(z))|0\rangle|_{z=0} = a_{(n)}(b(z)|0\rangle|_{z=0})$, where $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)}z^{-n-1}$, as usual.

11. (due 2/21) Prove that in any vertex algebra one has $Y(Ta, z) = \partial_z Y(a, z)$.

12. (due 2/28) Let A and B be two linear operators on a (possibly infinite-dimensional) vector space V . Prove that:

(a) $e^{xA} B e^{-xA} = e^{x \text{ad} A} B$, where both sides belong to $(\text{End } V)[[x]]$ and $(\text{ad } A)B = [A, B] = AB - BA$.

(b) $e^{xA} e^{yB} = e^{xy[A, B]} e^{yB} e^{xA}$, provided that $[A, B]$ commutes with A and B .

13. (due 2/28) Rewrite the Borcherds identity

$$Y(a, z)Y(b, w) i_{z,w}(z-w)^n - Y(b, w)Y(a, z) i_{w,z}(z-w)^n = \sum_{j \geq 0} Y(a_{(n+j)}b, w) \partial_w^{(j)} \delta(z, w)$$

in terms of modes.

14. (due 3/13) Let V be a vertex algebra, in which there are two odd elements, f_+ and f_- , with the property that $\psi^\pm(z) = Y(f_\pm, z)$ is a pair of charged free fermions. More precisely, we assume the following (anti)commutators:

$$[\psi^+(z), \psi^+(w)] = [\psi^-(z), \psi^-(w)] = 0, \quad [\psi^+(z), \psi^-(w)] = \delta(z, w).$$

For a fixed $\lambda \in \mathbb{C}$, let

$$a(z) = :\psi^+(z)\psi^-(z):, \quad L^\lambda(z) = (1 - \lambda) :(\partial\psi^+)(z)\psi^-(z): + \lambda :(\partial\psi^-)(z)\psi^+(z):.$$

Prove that:

- (a) $a(z)$ is a free boson field with level 1 and $[a(z), \psi^\pm(w)] = \pm\psi^\pm(w)\delta(z, w)$.
- (b) $L^\lambda(z)$ is a Virasoro field with central charge $c_\lambda = -12\lambda^2 + 12\lambda - 2$.
- (c) The field $\psi^+(z)$ (respectively, $\psi^-(z)$) is a primary field with respect to $L^\lambda(z)$ of conformal weight λ (respectively, $1 - \lambda$).

15.(due 3/13) Recall that the Casimir element for \mathfrak{sl}_n is defined as $\bar{\Omega} = \sum a^i b^i \in U(\mathfrak{sl}_n)$ where $\{a^i\}, \{b^i\}$ are dual bases with respect to the bilinear form $(a|b) = \text{tr } ab$. Prove that $\bar{\Omega}$ acts as multiplication by $2n$ in the adjoint representation.

16.(due 3/20) Consider a finite-dimensional simple Lie algebra \mathfrak{g} equipped with a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot)$, and let $\{a^i\}, \{b^i\}$ be dual bases of \mathfrak{g} with respect to $(\cdot|\cdot)$. Denote by h^\vee the dual Coxeter number of \mathfrak{g} , i.e., $1/2$ the eigenvalue of the Casimir element $\bar{\Omega} = \sum a^i b^i$ in the adjoint representation (cf. Problem 15). For an affine vertex algebra V associated to the affine Lie algebra $\hat{\mathfrak{g}}$ at level $k \neq -h^\vee$, define

$$\nu = \frac{1}{2(k + h^\vee)} \sum_i a_{(-1)}^i b_{(-1)}^i |0\rangle \in V.$$

Prove that $L(z) = Y(\nu, z)$ is a Virasoro field with central charge $c = k \dim \mathfrak{g} / (k + h^\vee)$.

17.(due 4/3) Let Q be an integral lattice with a bilinear form $(\cdot|\cdot)$, and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ with the bilinear form extended to \mathfrak{h} . Recall that the lattice vertex algebra V_Q is conformal with conformal vector

$$\nu = \frac{1}{2} \sum_i a_{(-1)}^i b_{(-1)}^i |0\rangle,$$

where $\{a^i\}, \{b^i\}$ are dual bases of \mathfrak{h} with respect to $(\cdot|\cdot)$. Prove that the vertex operator $Y(|\alpha\rangle, z)$ is a primary field of conformal weight $(\alpha|\alpha)/2$ for any $\alpha \in Q$.

18.(due 4/3) Recall that the Schur polynomials S_d have the generating function

$$\sum_{d=0}^{\infty} u^d S_d(x_1, x_2, x_3, \dots) = \exp\left(\sum_{m=1}^{\infty} x_m u^m\right).$$

Prove that

$$\sum_{n=0}^d \frac{\lambda^n}{n!} S_{d-n}(x_1, x_2, x_3, \dots) = S_d(\lambda + x_1, x_2, x_3, \dots).$$

19.(due 4/9) Show that the tensor product $V = V_1 \otimes V_2$ of two vertex algebras V_1, V_2 is a vertex algebra, and V is conformal if V_1, V_2 are conformal.

20. (*due 4/9*) Prove that for any element a in a vertex algebra V , the zero mode $a_{(0)}$ is a derivation of V .