1. a) 
\( y(t) = \) amount of radioactive material left after \( t \) days.
\( y'(t) = \) rate of change (decrease) of \( y(t) \).
The rate of change of \( y(t) \) is proportional to \( y(t) \):
\[ y' = -\lambda y, \]
where \( \lambda > 0 \) is the decay constant. All solutions of this differential equation have the form
\[ y(t) = Ce^{-\lambda t}, \quad C = \text{const}. \]
We have
\[ y(0) = 6, \quad y(50) = 2, \]
which gives \( C = 6, \quad Ce^{-50\lambda} = 2 \). From here we find \( e^{-50\lambda} = 1/3 \approx .33333, -50\lambda = \ln(1/3), \quad \lambda = -\ln(1/3)/50 \approx .022 \).

1. b) We want to find \( t \) such that \( y(t) = 1 \). Then \( 6e^{-\lambda t} = 1, \quad e^{-\lambda t} = 1/6 \approx .16667, \quad -\lambda t = \ln(1/6), \quad t = \frac{\ln(1/6)}{\ln(1/3)/50} \approx 81 \).

2. Note: Here I only give directions how to sketch the graphs; I will sketch them in class.

2. a) Draw two coordinate systems, \( yz \) and \( ty \).

In the \( yz \)-plane sketch the graph of the function \( z = 8 - 2y \). This is a linear function, so its graph is a straight line. It passes through the points (3, 2) and (4, 0). The solution to the equation \( 8 - 2y = 0 \) is \( y = 4 \).

In the \( ty \)-plane we’ll sketch the solutions of \( y' = 8 - 2y \). There is one constant solution, \( y = 4 \). Every non-constant solution will either increase or decrease all the time. It will be either unbounded or asymptotic to a constant solution.
(i) The solution with initial condition $y(0) = 2$ passes through the point $(0, 2)$ in the $ty$-plane. When $y = 2$, we have $z = 8 - 2y = 4 > 0$; hence this solution is increasing. When it increases from $y = 2$ it cannot cross the constant solution $y = 4$, so it will be asymptotic to it. It is concave down, because when $y$ increases from 2 to 4, $z = 8 - 2y$ decreases.

(ii) The solution with initial condition $y(0) = 4$ is the constant solution $y = 4$ and its graph is a straight line.

2. b) Draw two coordinate systems, $yz$ and $ty$.

In the $yz$-plane sketch the graph of the function $z = -(y + 3)(y - 5) = -y^2 + 2y + 15$. This is a quadratic function, so its graph is a parabola. The equation $z = 0$ has two solutions, $y = -3$ and $y = 5$. The parabola is concave down because the coefficient in front of $y^2$ is negative. It passes through the points $(-3, 0)$ and $(5, 0)$. The vertex of the parabola is in the middle between the zeroes, at $y = (-3 + 5)/2 = 1$.

In the $ty$-plane we’ll sketch the solutions of $y' = -(y + 3)(y - 5)$. There are two constant solutions, $y = -3$ and $y = 5$. Every non-constant solution will either increase or decrease all the time. It will be either unbounded or asymptotic to a constant solution.

(i) The solution with initial condition $y(0) = -4$ passes through the point $(0, -4)$ in the $ty$-plane. When $y = -4$, we have $z < 0$; hence this solution is decreasing. When $y$ decreases from $-4$ to $-\infty$, $z$ will decrease, hence the solution is concave down. It is unbounded because it doesn’t encounter a constant solution.

(ii) The solution with initial condition $y(0) = 2$ passes through the point $(0, 2)$ in the $ty$-plane. When $y = 2$, we have $z > 0$; hence this solution is increasing. When $y$ increases from 2 to 5, $z$ decreases, and the solution is concave down. It is asymptotic to the line $y = 5$.

(iii) The solution with initial condition $y(0) = 6$ passes through the point $(0, 6)$ in the $ty$-plane. When $y = 6$, we have $z < 0$; hence this solution is decreasing. When $y$ decreases from 6 to 5, $z$ increases, hence the solution is concave up. It is asymptotic to the line $y = 5$. 
3. a) We use separation of variables (note that \(e^{2y} \neq 0\)).

\[
y' = te^{2y}, \quad \frac{y'}{e^{2y}} = t, \quad e^{-2y}y' = t,
\]

\[
\int e^{-2y}y'dt = \int t\,dt + C, \quad \int e^{-2y}dy = \int t\,dt + C,
\]

\[
e^{-2y} = \frac{t^2}{2} + C, \quad e^{-2y} = -t^2 - 2C,
\]

\[
-2y = \ln(-t^2 - 2C), \quad y = -\frac{1}{2}\ln(-t^2 - 2C), \quad C = \text{const}.
\]

3. b) We first solve the differential equation, using separation of variables. Note that \(y = 0\) is a constant solution but it doesn’t satisfy the initial condition \(y(0) = 1\). Thus, we can assume \(y \neq 0\) and we can divide by \(y^2\).

\[
y' = y^2 - e^{3t}y^2, \quad \frac{y'}{y^2} = 1 - e^{3t}, \quad y^{-2}y' = 1 - e^{3t},
\]

\[
\int y^{-2}y'dt = \int (1 - e^{3t})dt + C, \quad \int y^{-2}dy = \int (1 - e^{3t})dt + C,
\]

\[
\frac{y^{-2+1}}{-2 + 1} = t - \frac{e^{3t}}{3} + C, \quad -y^{-1} = t - \frac{e^{3t}}{3} + C, \quad -\frac{1}{y} = \frac{3t - e^{3t} + 3C}{3},
\]

\[
y = -\frac{3}{3t - e^{3t} + 3C}.
\]

Then we impose the initial condition \(y(0) = 1\) and we get an equation for \(C\):

\[
1 = -\frac{3}{3\cdot0 - e^{3\cdot0} + 3C} = -\frac{3}{-1 + 3C},
\]

which gives \(-1 + 3C = -3\), \(3C = -2\), \(C = -\frac{2}{3}\).

Answer: \(y(t) = -\frac{3}{3t - e^{3t} - 2}\).
4. a) 
\[ f(t) = \text{number of fish in the pond at time } t. \]
\[ f'(t) = \text{rate of change of } f(t). \]
Growth constant = 3% − 1% = .03 − .01 = .02.
The rate of change is proportional to \( f(t) \):
\[ f'(t) = .02f(t). \]

This means that \( f(t) \) is a solution of the differential equation \( y' = .02y \).

4. b) If \( M \) fish are harvested each month, we subtract \( M \) from the rate of change of the number of fish. We get
\[ g'(t) = .02g(t) - M, \]
so \( g(t) \) is a solution of the differential equation \( y' = .02y - M \).

The function \( z(y) = .02y - M \) is linear and it has only one zero, namely \( y = M/.02 = 50M \). The line \( y = 50M \) is a constant solution to our differential equation. We are interested in the solution with an initial condition \( y(0) = 5000 \). This solution passes through the point \((0, 5000)\) in the \( ty \)-plane.

When \( y = 5000 \), the corresponding \( z \) is \( .02 \times 5000 - M = 100 - M \). If \( 100 - M > 0 \), our solution is increasing. When \( 100 - M = 0 \), it is constant, and when \( 100 - M < 0 \), it is decreasing. We don’t want \( y \) to be decreasing, because in this case it will reach \( y = 0 \) and all the fish will die out. So we must have \( 100 - M \geq 0 \), or \( M \leq 100 \).

Thus the maximal \( M \) is 100. For this value of \( M \), the solution with an initial condition \( y(0) = 5000 \) is constant.