Lie Theory and Its Applications in Physics VII
ed. V.K. Dobrev et al, Heron Press, Sofia, 2008

Constructing models of vertex algebras in higher dimensions

Bojko Bakalov[1], **Nikolay M. Nikolov**[2]

[1]Department of Mathematics, North Carolina State University,
Box 8205, Raleigh, NC 27695, USA;
bojko_bakalov@ncsu.edu

[2]Institute for Nuclear Research and Nuclear Energy,
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria;
mitov@inrne.bas.bg

Abstract

Vertex algebras in higher dimensions correspond to models of Quantum Field Theory (Wightman axioms) with Global Conformal Invariance. We review how such a vertex algebra can be generated from a collection of local fields, or from a vertex Lie algebra. The one-dimensional restriction of a vertex algebra in higher dimensions to a time-like line gives a chiral vertex algebra endowed with an action of the (higher-dimensional) conformal Lie algebra. We announce that from such data one can reconstruct the initial vertex algebra.

1 Definition of a vertex algebra on $\mathbb{C}^D$

One of the main problems of mathematical physics, which has remained open for more than fifty years, is to construct models of interacting quantum fields in higher dimensions. In the presence of the so-called global conformal invariance [14] this problem becomes purely algebraic. The corresponding algebraic structures are vertex algebras in higher dimensions [13] (see also [17] for an earlier work and [9] for a more general definition). These algebras appear as a straightforward generalization of the vertex algebras from two-dimensional conformal field theories, first introduced axiomatically in [8] (see [11] for an excellent introduction to the subject).

A (boson) *vertex algebra* (on $\mathbb{C}^D$) consists of the following data [13]: a vector space V (the space of states), a state-field correspondence (defined below), infinitesimal translations $T_\mu \in \mathrm{End} V$ ($\mu = 1, \dots, D$), and a vector $\mathbf{1} \in V$ (vacuum). These data have to satisfy certain conditions, or axioms, which are called: locality, translation covariance, and the vacuum axiom. We will now explain the definition in more detail, comparing the cases when the space-time dimension $D = 1$ and $D > 1$.

The *state-field correspondence* for $D = 1$ is given by a linear map

$$\begin{aligned} V \otimes V \; &\rightarrow \; V[[z]]\,[1/z]\,, \\ a \otimes b \; &\mapsto \; a(z)\,b \; = \sum_{n=-N_{ab}}^{\infty} a_{\{n\}}b\; z^n\,. \end{aligned}$$

Here, z is single complex variable and $a(z)b$ is a formal power series in z containing negative powers of z that are bounded from below by a number N_{ab} depending on a and b. The physical meaning of z is that it is a complexified light-cone variable. In the physical space z takes values in the punctured unit circle $U(1)\backslash\{-1\} \subseteq \mathbb{C}$.

In the case $D > 1$ the state-field correspondence is given by a linear map

$$\begin{aligned} V \otimes V \; &\rightarrow \; V[[\mathbf{x}]]\,[1/\mathbf{x}^2]\,, \\ a \otimes b \; &\mapsto \; a(\mathbf{x})\,b = \sum_{\substack{n \geqslant -N_{ab} \\ \sigma\,, m \geqslant 0}} a_{\{n,m,\sigma\}}b\;(\mathbf{x}^2)^n\,h_{m,\sigma}(\mathbf{x})\,, \end{aligned}$$

where $\mathbf{x} = (x^1, \dots, x^D)$ is a D-dimensional formal complex variable and

$$\mathbf{x}^2 := (x^1)^2 + \cdots + (x^D)^2\,.$$

The series $a(\mathbf{x})b$ can be uniquely expanded by using a basis of homogeneous harmonic polynomials $\{h_{m,\sigma}(\mathbf{x})\}_\sigma$ of degree $m = 0, 1, \dots$. The variable $\mathbf{x}$ is related by a complex conformal transformation to the complexified space-time coordinates of the Minkowski space. A basic property of this change of coordinates is that the real Minkowski space is mapped onto a bounded subset in $\mathbb{C}^D$ with closure $U(1)\,\mathbb{S}^{D-1}$, the *compactified Minkowski space* [17, 13]. A linear map from V to $V[[\mathbf{x}]]\,[1/\mathbf{x}^2]$ is called a *field* on V; thus, a state-field correspondence is equivalent to a map $a \mapsto a(\mathbf{x})$ from the space of states V to the space of fields on V.

To explain the axioms of a vertex algebra, we begin with the *locality* in the case $D = 1$, which reads:

$$(z-w)^N\,a(z)\,b(w)\,c = (z-w)^N\,b(w)\,a(z)\,c\,,$$

where both sides belong to the space $V[[z,w]]\,[1/z, 1/w]$. In this case, we also say that the fields $a(z)$ and $b(z)$ are *mutually local*.

The locality for $D > 1$ is formulated by replacing the “distance” $z - w$ with the interval $(\mathbf{x}-\mathbf{y})^2$:

$$\big((\mathbf{x}-\mathbf{y})^2\big)^N a(\mathbf{x})\,b(\mathbf{y})\,c = \big((\mathbf{x}-\mathbf{y})^2\big)^N b(\mathbf{y})\,a(\mathbf{x})\,c\,,$$

the equality being understood in the space $V[[\mathbf{x},\mathbf{y}]]\,[1/\mathbf{x}^2, 1/\mathbf{y}^2]$.

The *translation covariance* and the *vacuum axiom* for $D = 1$ are:

$$[T, a(z)] = \frac{d}{dz}\, a(z)\,, \qquad \mathbf{1}(z)\, a = a\,, \qquad a(z)\,\mathbf{1}|_{z=0} = a\,,$$

while in the case $D > 1$ they read:

$$[T_\mu, a(\mathbf{x})] = \frac{\partial}{\partial x^\mu}\, a(\mathbf{x})\,, \qquad \mathbf{1}(z)\, a = a\,, \qquad a(\mathbf{x})\,\mathbf{1}|_{\mathbf{x}=0} = a\,.$$

The $D = 1$ vertex algebras are also called *chiral vertex algebras*.

2 Conformal and hermitian vertex algebras, and their relation to GCI QFT

In order to have a passage from vertex algebras to quantum fields, we need some additional structures on the former, namely, the conformal and hermitian structures introduced below.

Consider the complex *conformal Lie algebra* $\mathfrak{c}_D$ with generators T_μ, C_μ, H, $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$ for $\mu, \nu = 1, \dots, D$, and relations

$$\begin{gathered}
[H, \Omega_{\mu\nu}] = [T_\mu, T_\nu] = [C_\mu, C_\nu] = 0\,, \\
[\Omega_{\mu\nu}, T_\gamma] = \delta_{\mu\gamma} T_\nu - \delta_{\nu\gamma} T_\mu\,, \quad [\Omega_{\mu\nu}, C_\gamma] = \delta_{\mu\gamma} C_\nu - \delta_{\nu\gamma} C_\mu\,, \\
[H, T_\mu] = T_\mu\,, \quad [H, C_\mu] = -C_\mu\,, \quad [T_\mu, C_\nu] = 2\,\delta_{\mu\nu} H - 2\,\Omega_{\mu\nu}\,, \\
[\Omega_{\mu_1\nu_1}, \Omega_{\mu_2\nu_2}] = \delta_{\mu_1\mu_2}\,\Omega_{\nu_1\nu_2} + \delta_{\nu_1\nu_2}\,\Omega_{\mu_1\mu_2} - \delta_{\mu_1\nu_2}\,\Omega_{\nu_1\mu_2} - \delta_{\nu_1\mu_2}\,\Omega_{\mu_1\nu_2}\,.
\end{gathered}$$

Its real form is fixed by the following Hermitian conjugation:

$$H^* \,=\, -H\,, \quad \Omega^*_{\mu\nu} \,=\, \Omega_{\mu\nu}\,, \quad T^*_\mu \,=\, C_\mu\,.$$

In a conformal vertex algebra the vector space of states is a $\mathfrak{c}_D$-module. We will consider only $\mathfrak{c}_D$-modules that satisfy the following two properties: H is diagonalizable with non-negative eigenvalues (*energy positivity*), and every vector is contained in a finite-dimensional subspace invariant under all $\Omega_{\alpha\nu}$ (*integrability*). A vertex algebra V is called *conformal* [13] if it is equipped with an action of $\mathfrak{c}_D$, which annihilates the vacuum $\mathbf{1}$ and satisfies

$$\begin{aligned}
[H, a(\mathbf{x})] \,&=\, (H\,a)(\mathbf{x}) + \mathbf{x}\cdot\partial_{\mathbf{x}}\, a(\mathbf{x}), \\
[\Omega_{\mu\nu}, a(\mathbf{x})] \,&=\, (\Omega_{\mu\nu}\, a)(\mathbf{x}) + (x^\mu \partial_{x^\nu} - x^\nu \partial_{x^\mu})\, a(\mathbf{x}), \\
[C_\mu, a(\mathbf{x})] \,&=\, (C_\mu a)(\mathbf{x}) - 2\, x^\mu\, (H\, a)(\mathbf{x}) - 2 \sum_\nu x^\nu\, (\Omega_{\mu\nu}\, a)(\mathbf{x}) \\
&\quad + (\mathbf{x}^2\, \partial_{x^\mu} - 2\, x^\mu\, \mathbf{x}\cdot\partial_{\mathbf{x}})\, a(\mathbf{x}).
\end{aligned}$$

If we restrict the above definition to the case $D = 1$, then the resulting structure is called a *Möbius* vertex algebra (see [11]).

An *hermitian* vertex algebra [13] is a conformal vertex algebra endowed with an Hermitian nondegenerate form $\langle\cdot,\cdot\rangle$ and a TCP operator $a \mapsto a^+$ on V, such that

$$\langle a,\, c^+(\mathbf{x})\, b\rangle = \langle\, (\pi_{\mathbf{x}}\, c)\Big(\frac{R_D\,\mathbf{x}}{\mathbf{x}^2}\Big)\, a,\, b\,\rangle\,.$$

Here the map $\pi_{\mathbf{x}} \colon V \to V[\mathbf{x}]\,[1/\mathbf{x}^2]$ is given by

$$\pi_{\mathbf{x}} := e^{R_D \mathbf{x} \cdot \mathrm{C}} \, (\mathbf{x}^2)^{-H} \, \mathrm{Rot}(R_D \, \mathbf{x}, -\mathbf{e}_D) \,,$$

where $\mathrm{C} := (C_1, \dots, C_D)$,

$$R_D \mathbf{x} := (x^1, \dots, x^{D-1}, -x^D) \,, \quad \mathbf{e}_D := (0, \dots, 0, 1) \,,$$

and $\mathrm{Rot}(R_D \, \mathbf{x}, -\mathbf{e}_D)$ is a rotation in the plane spanned by the vectors $R_D \, \mathbf{x}$ and $-\mathbf{e}_D$ mapping the first one to the second. Finally, an hermitian vertex algebra is called *unitary* if the Hermitian form $\langle \cdot , \cdot \rangle$ is positive definite.

Theorem 1 ([13]). *There is a one-to-one correspondence between unitary vertex algebras* (*on* $\mathbb{C}^D$) *and models of Wightman axioms with global conformal invariance.*

3 Reconstruction theorems

The following theorems are the main tools for constructing vertex algebras. The first one is an analog of the Kac existence theorem [11] in higher dimensions.

Theorem 2 ([13, 2, 4]). *Let* V *be a vector space and* F *be a subspace of the space of fields on* V*, so that we have a linear map* (*field action*) $F \otimes V \to V[[\mathbf{x}]]\,[1/\mathbf{x}^2]$*. Assume that we are given actions of* $\mathfrak{c}_D$ *on* F *and* V*, and a vector* $\mathbf{1} \in V$*, satisfying the following properties*: *all fields from* F *are mutually local, the field action of* F *on* V *is* $\mathfrak{c}_D$*-equivariant, the vacuum* $\mathbf{1}$ *is* $\mathfrak{c}_D$*-invariant, and the modes of fields from* F *acting on* $\mathbf{1}$ *generate* V*. Then there exists a unique structure of a conformal vertex algebra on* V*, together with an embedding* $F \hookrightarrow V$ *that agrees with the state-field correspondence on* V*.*

The above theorem can be stated succinctly that a vertex algebra V can be generated by a collection F of local fields on V. If, in addition, we have equivariant actions of the conformal Lie algebra $\mathfrak{c}_D$, then the vertex algebra is conformal.

Note that the locality condition depends only on the commutators of fields. In [2] we investigated the algebraic structure obeyed by the commutators of fields in a vertex algebra. In particular, we showed that the commutator of two fields $a(\mathbf{x})$, $b(\mathbf{y})$ is determined by the *singular* (in $\mathbf{x}^2$) *part* of the field $a(\mathbf{x})$. We derived certain skew-symmetry and Jacobi identities satisfied by the singular parts of fields, thus obtaining the notion of a *vertex Lie algebra*. For $D = 1$ the corresponding notion was introduced by V.G. Kac (under the name Lie conformal algebra), and was intensely studied by Kac and collaborators (see, e.g., [11, 10, 1]). It turns out that the chiral vertex algebras corresponding to highest-weight representations of the Virasoro or affine Kac–Moody algebras can be generated by certain ($D = 1$) vertex Lie algebras (see [11]). We hope that the higher-dimensional analog will also lead to interesting examples.

Closely related to vertex Lie algebras are the so-called *Lie fields' models*. Lie fields are (Huygens) local fields whose modes span a Lie algebra. In particular, any chiral ($D = 1$) vertex algebra provides a system of Lie fields. For $D > 2$ the only known examples of Lie fields are the free ones, and there is a general negative result for a *single scalar* field (in $D > 2$) [15, 12, 7]. However, there are examples of Lie polylocal fields, e.g., Wick products of free fields $:\varphi(\mathbf{x})\varphi(\mathbf{y}):$. More general bilocal fields whose modes span infinite-dimensional Lie algebras and their unitary positive-energy representations are investigated in [5, 6].

The next theorem is an algebraic version of the Wightman reconstruction theorem [16]; its proof is completely analogous and is therefore omitted.

Theorem 3. *An hermitian vertex algebra V can be uniquely constructed from a vector space F, an action of $\mathfrak{c}_D$ on F, an anti-linear involution $a \mapsto a^+$ on F, and a system of rational functions* $(a_k \in F)$

$$W_N(\mathbf{x}_1, a_1; \dots; \mathbf{x}_N, a_N) \in \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_N]\left[1/\prod_{j<k}(\mathbf{x}_j - \mathbf{x}_k)^2\right],$$

such that W_N are symmetric (*under simultaneous permutations of the* $\mathbf{x}$*'s and* a*'s*), *$\mathfrak{c}_D$-invariant, and TCP-invariant.*

The $\mathfrak{c}_D$-invariance in the above theorem can be spelled out as

$$\Big(\sum_{k=1}^{N} \partial_{x_k^\mu}\Big) W_N = 0\,, \qquad \sum_{k=1}^{N}\big(\mathbf{x}_k \cdot \partial_{\mathbf{x}_k} W_N + W_N(\mathbf{x}_k, H\,a_k)\big) = 0\,, \quad \text{etc.,}$$

while the TCP-invariance means

$$W_N(\mathbf{x}_k, a_k) = W_N(-\overline{\mathbf{x}}_k, a_k^+)\,.$$

(In the above formulas we have suppressed some of the arguments of W_N.)

4 Restriction to lower dimensions and reconstruction from it

If V is a vertex algebra in dimension D, then we define its *restriction* to spacetime dimension 1 as a chiral vertex algebra with the same space of states V and vacuum $\mathbf{1}$, with a translation endomorphism T_1, and a state-field correspondence given by

$$a(z)\,b \;\equiv\; Y_1(a,z)\,b \;:=\; a(\mathbf{x})\,b\,\big|_{\mathbf{x}\,=\,(z,0,\dots,0)}\,.$$

Such a chiral vertex algebra will be denoted as $(V, \mathbf{1}, T_1, Y_1)$. Recall that a Möbius vertex algebra is equipped with an action of the Möbius Lie algebra $\mathfrak{c}_1 \equiv \mathfrak{sl}_2$ (see Sect. 2 and [11]).

Theorem 4 ([4]). *Let $(V, \mathbf{1}, \mathfrak{c}_1, Y_1)$ be a unitary Möbius* (*chiral*) *vertex algebra, equipped with an action of $\mathfrak{c}_D$, which extends the action of $\mathfrak{c}_1$, annihilates the vacuum $\mathbf{1}$, and satisfies* $(\mu, \nu = 2, \dots, D)$:

$$\begin{gathered}[T_\mu, a(z)] = (T_\mu\, a)(z)\,, \qquad [\Omega_{\mu\nu}, a(z)] = (\Omega_{\mu\nu}\, a)(z)\,, \\ [\Omega_{1\mu}, a(z)] = (\Omega_{1\mu}\, a)(z) + z\,(T_\mu a)(z)\,, \\ [C_\mu, a(z)] = (C_\mu a)(z) + 2\,z\,(\Omega_{1\mu}\, a)(z) + z^2\,(T_\mu a)(z)\,.\end{gathered}$$

Then V can be endowed with a unique structure of a unitary vertex algebra in dimension D, so that $(V, \mathbf{1}, \mathfrak{c}_1, Y_1)$ is its restriction to dimension 1.

This theorem was announced (in a different form) at the International Congress of Mathematicians in August 2006; see [3]. It will be proved in [4].

5 Conclusions

- Globally conformal invariant QFT can be a candidate for the theory of observable fields in conformal field theories.
- It has a purely algebraic description, which generalizes the notion of a chiral vertex algebra.
- The singular part of the operator product expansion closes a substructure called a vertex Lie algebra.
- A unitary Möbius chiral vertex algebra, for which the action of the Möbius Lie algebra is extended to an action of the higher-dimensional conformal Lie algebra, generates a GCI QFT in higher dimensions.

Acknowledgments

N.M.N. acknowledges the invitation to the Varna meeting, where this paper was presented. B.B. was partially supported by NSF grant DMS-0701011. N.M.N. was supported in part by the Research Training Network of the European Commission under contract MRTN-CT-2004-00514 and by the Bulgarian National Council for Scientific Research under contract PH-1406.

References

[1] B. Bakalov, A. D'Andrea and V.G. Kac, Theory of finite pseudoalgebras, *Adv. Math.* **162**, 1–140 (2001).

[2] B. Bakalov and N.M. Nikolov, Jacobi identity for vertex algebras in higher dimensions, *J. Math. Phys.* **47**, 053505 (2006); math-ph/0604069.

[3] B. Bakalov and N.M. Nikolov, Reconstruction of a vertex algebra in higher dimensions from its one-dimensional restriction, in: Abstracts of Short Communications at ICM 2006, Madrid; http://www.icm2006.org/v_f/AbsDef/Shorts/abs_1622.pdf

[4] B. Bakalov and N.M. Nikolov, in preparation.

[5] B. Bakalov, N.M. Nikolov, K.-H. Rehren and I.T. Todorov, Unitary positive-energy representations of scalar bilocal quantum fields, *Commun. Math. Phys.* **271**, 223–246 (2007); math-ph/0604069.

[6] B. Bakalov, N.M. Nikolov, K.-H. Rehren and I.T. Todorov, Infinite dimensional Lie algebras in 4D conformal quantum field theory, in these proceedings; arXiv:0711.0627.

[7] K. Baumann, There are no scalar Lie fields in three or more dimensional space-time, *Commun. Math. Phys.* **47**, 69–74 (1976).

[8] R.E. Borcherds, Vertex algebras, Kac–Moody algebras, and the Monster, *Proc. Nat. Acad. Sci. U.S.A.* **83**, 3068–3071 (1986).

[9] R.E. Borcherds, Vertex algebras, in: M. Kashiwara, et al. (eds.), *Topological field theory, primitive forms and related topics*, Progr. Math. **160**, pp. 35–77, Birkhäuser Boston, Boston, MA, 1998.

[10] A. D'Andrea and V.G. Kac, Structure theory of finite conformal algebras, *Selecta Math. (N.S.)* **4**, 377–418 (1998).

[11] V.G. Kac, *Vertex algebras for beginners*, 2nd edition, University Lecture Series **10**, Amer. Math. Soc., Providence RI, 1998.

[12] J.H. Lowenstein, The existence of scalar Lie fields, *Commun. Math. Phys.* **6**, 49–60 (1967).

[13] N.M. Nikolov, Vertex algebras in higher dimensions and globally conformal invariant quantum field theory, *Commun. Math. Phys.* **253**, 283–322 (2005).

[14] N.M. Nikolov and I.T. Todorov, Rationality of conformally invariant local correlation functions on compactified Minkowsi space, *Commun. Math. Phys.* **218**, 417–436 (2001), hep-th/0009004.

[15] D.W. Robinson, On a soluble model of relativistic field theory, *Phys. Lett.* **9**, 189–191 (1964).

[16] R.F. Streater and A.S. Wightman, *PCT, spin and statistics, and all that*, W.A. Benjamin, Inc., New York–Amsterdam, 1964.

[17] I.T. Todorov, Infinite dimensional Lie algebras in conformal QFT models, in: A.O. Barut, H.-D. Doebner (eds.), *Conformal groups and related symmetries, physical results and mathematical background*, Lecture Notes in Phys. **261**, pp. 387–443, Springer–Verlag, Berlin, 1986.