Chapter 6

Discrete Probability Distributions

Random variables are frequently grouped into different families depending upon the characteristics of their distribution functions. For example, in previous chapters we alluded to the family of “geometric random variables” and the family of “exponential random variables”, among others. The distribution functions of all random variables that belong to a given family differ only in the value assigned to a small number of parameters. For example, the probability mass function of a discrete random variable may be given by $0.5 \times 0.5^{n-1}$, for $n = 1, 2, \ldots$, while for another it may be $0.75 \times 0.25^{n-1}$, for $n = 1, 2, \ldots$, and zero otherwise. Both these random variables belong to the family of (geometric) random variables whose probability mass function is written as

$$p_X(n) = \begin{cases} p(1-p)^{n-1}, & n = 1, 2, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

for $0 < p < 1$: They differ only in the value assigned to the parameter $p$. In this chapter we consider the most common families of discrete random variables; in the next, we turn our attention to families of continuous random variables.

6.1 The Discrete Uniform Distribution

Let $X$ be a discrete random variable that can assume the values $\{x_1, x_2, \ldots, x_n\}$. The discrete uniform distribution is obtained when each of these values has equal probability of occurring (all outcomes are equally likely) and, since there are $n$ of them, each must have probability equal to $1/n$. The discrete uniform probability mass function is given as

$$p_X(x_i) = \text{Prob}\{X = x_i\} = \begin{cases} 1/n, & i = 1, 2, \ldots, n, \\ 0, & \text{otherwise}. \end{cases}$$
The corresponding cumulative distribution function is
\[ F_X(t) = \sum_{i=1}^{\lfloor t \rfloor} p_X(x_i) = \frac{\lfloor t \rfloor}{n} \quad \text{for } 1 \leq t \leq n, \]
while \( F_x(t) = 0 \) for \( t < 1 \) and \( F_X(t) = 1 \) for \( t > n \).

**Example 6.1** An example of a discrete uniform random variable is the random variable that denotes the number of spots that appear when a fair die is thrown once. Its probability mass function is given by \( p(x_i) = \frac{1}{6} \) if \( x_i \in \{1, 2, \ldots, 6\} \) and is equal to zero otherwise.

When the \( x_i \) are equally spaced, as in the previous example, a bar graph of the probability mass function of this distribution will show \( n \) equally spaced bars all of height \( \frac{1}{n} \), while a plot of the cumulative distribution function will display a perfect staircase function of \( n \) identical steps, each of height \( \frac{1}{n} \), beginning at zero just prior to the first step at \( x_1 \) and rising to the value 1 at the last step at \( x_n \). In the specific case in which \( x_i = i \) for \( i = 1, 2, \ldots, n \), the moments of the discrete uniform distribution are given by
\[ E[X^k] = \sum_{i=1}^{n} \frac{i^k}{n}. \]
In particular,
\[ E[X] = \sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n + 1)}{2} = \frac{n + 1}{2}, \]
and, using the well-known formula for the sum of the squares of the first \( n \) integers,
\[ E[X^2] = \sum_{i=1}^{n} \frac{i^2}{n} = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \left( \frac{n(n + 1)(2n + 1)}{6} \right) = \frac{(n + 1)(2n + 1)}{6}. \]
The variance is then computed as
\[ \text{Var}[X] = E[X^2] - (E[X])^2 = \frac{(n + 1)(2n + 1)}{6} - \left( \frac{n + 1}{2} \right)^2 = \frac{n^2 - 1}{12}. \]
When the \( x_i \) are spaced at unit distance apart but begin at \( x_1 = a \) and end at \( x_n = b = a + n - 1 \), the corresponding formulae for the mean and variance (using \( b - a = n - 1 \)) are respectively
\[ E[X] = \frac{a + b}{2}, \quad \text{Var}[X] = \frac{(b - a + 2)(b - a)}{12} = \frac{n^2 - 1}{12}. \]
The probability generating function for a discrete uniformly distributed random variable \( X \) is
\[ G_X(z) = \sum_{i=1}^{n} \frac{z^i}{n} = \frac{1}{n} \sum_{i=1}^{n} z^i. \]
Its moment generating function, $M_X(\theta)$, is obtained by setting $z = e^\theta$, since $M_X(e^\theta) = G_X(e^\theta)$. Thus
\[
M_X(\theta) = \frac{1}{n} \sum_{i=1}^{n} e^{\theta i} \quad \text{and} \quad M_X(0) = \frac{1}{n} \sum_{i=1}^{n} e^{0} = 1.
\]

Worked problems

Problem 6.1.1 The number of letters, $X$, delivered to our home each day is uniformly distributed between 3 and 10. Find

(a) The probability mass function of $X$.
(b) $\text{Prob}\{X < 8\}$.
(c) $\text{Prob}\{X > 8\}$.
(d) $\text{Prob}\{2 \leq X \leq 5\}$.

Answer 6.1.1

(a) $p_X(k) = \begin{cases} 1/8, & k = 3, 4, \ldots, 10, \\ 0, & \text{otherwise}. \end{cases}$

(b) $\text{Prob}\{X < 8\} = \sum_{k=3}^{7} p_X(k) = 5/8.$

(c) $\text{Prob}\{X > 8\} = \sum_{k=9}^{10} p_X(k) = 2/8.$

(d) $\text{Prob}\{2 \leq X \leq 5\} = \sum_{k=2}^{5} p_X(k) = 3/8.$

Problem 6.1.2 A random variable $X$ is uniformly distributed on the integers from 5 through 12.

(a) Find the expectation, variance, and standard deviation of $X$.
(b) What is the probability of $X$ lying within one standard deviation of its mean?

Answer 6.1.2

(a) $E[X] = \frac{5 + 12}{2} = 8.5$; $\text{Var}(X) = \frac{n^2 - 1}{12} = \frac{63}{12} = 5.25$; $\sigma = \sqrt{5.25} = 2.2913$.

(b) $\text{Prob}\{E[X] - \sigma \leq X \leq E[X] + \sigma\} = \text{Prob}\{8.5 - 2.2913 \leq X \leq 8.5 + 2.2913\}$

$= \text{Prob}\{6.2087 \leq X \leq 10.7913\}$

$= \text{Prob}\{7 \leq X \leq 10\} = 4/8.$