Test 4 Review

General Things

- 100 points possible
- NO CALCULATOR of any kind will be needed or allowed
- covers Chapter 13: Sections 13.1 - 13.6, Section 10.5, Section 12.6
- Extra Credit is available and will be due on Monday, April 21, 2008

Section 13.1

- A **vector field** is a function $\mathbf{F}$ that assigns to each point (x, y, z) a vector $\mathbf{F}(x, y, z)$
- ∇f is really a vector field and is called a **gradient vector field**.

Section 13.2

- Let C be a smooth curve defined by parametric equations
  - Divide curve into subarcs and form Riemann sum.
  - $\displaystyle\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
- Let C be defined by the vector equation $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
  - $\displaystyle\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$
  - $\displaystyle\int_C f(x,y,z)ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a continuous vector field.
  - $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt$

Section 13.3

- The Fundamental Theorem for Line Integrals
  - $\displaystyle\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
  - Evaluate the line integral of a conservative vector field, simply by knowing the value of f at the endpoints of C.
- If $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

- $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ iff **F** is a conservative vector field.

Section 13.5

- curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k}$
- If **F** is conservative, then curl $\mathbf{F} = \mathbf{0}$
- If **F** is a vector field and curl $\mathbf{F} = 0$, then **F** is a conservative vector field.
- The **divergence of F** is defined by div$\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$
- Vector Forms of Green's Theorem
  - $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\mathbf{F}) \cdot \mathbf{k} dA$
  - $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div}\mathbf{F}(x, y) dA$

Section 10.5

- Let $\mathbf{r}(u, v) = x(u, v)\ \mathbf{i} + y(u, v)\ \mathbf{j} + z(u, v)\ \mathbf{k}$ be a vector-valued function defined on a region D in the $uv-$ plane.
- In general, a surface given as the graph of a function of x and y, that is, with an equation of the form $z = f(x, y)$ can always be regarded as a parametric surface by taking x and y as parameters.
- tangent vectors
  - $\mathbf{r}_v = \dfrac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \dfrac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \dfrac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$
  - $\mathbf{r}_v = \dfrac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \dfrac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \dfrac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$

Section 12.6

- The **surface area** of S is $A(S) = \displaystyle\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$ where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$
- If S is $z = f(x, y)$ then $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$

Section 13.6

- The **surface integral** is defined as $\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|dA$
- Any surface with equation $z = g(x,y)$ has surface area
  - $\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$

Practice Problems

1. Find the gradient vector field of $f(x,y) = \frac{1}{4}(x+y)^2$.

2. Evaluate the line integral $\int_C xy^3 ds$ where C is given by the curve $x = 4\sin t, y = 4\cos t, z = 3t, 0 \leq t \leq \pi/2$

3. Evaluate the line integral $\mathbf{F}(x,y) = x^2y^3\mathbf{i} - y\sqrt{x}\mathbf{j}$ where C is given by the vector equation $\mathbf{r}(t) = t^2\mathbf{i} - t^3\mathbf{j}, 0 \leq t \leq 1$.

4. Determine whether $\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

5. Find a function f such that $\mathbf{F} = \nabla f$ and then evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and C is the line segment from (1,0,-2) to (4,6,3).

6. Find (a) the curl and (b) the divergence of the vector field $\mathbf{F}(x,y,z) = \mathbf{i} + (x + yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$.

7. Determine whether or not $\mathbf{F}(x,y,z) = ye^{-x}\mathbf{i} + e^{-x}\mathbf{j} + 2z\mathbf{k}$ is conservative.

8. Evaluate $\iint_S y dS$ where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$.

Answers

1. $\nabla f(x,y) = \left\langle \frac{1}{2}(x+y), \frac{1}{2}(x+y) \right\rangle$

2. 320

3. $\dfrac{-59}{105}$

4. $\mathbf{F}$ is conservative $f(x,y) = ye^x + x\sin y + K$

5. $f(x,y,z) = xyz + z^2 + K$ and $\int_C \mathbf{F} \cdot d\mathbf{r} = 77$

6. (a) curl $\mathbf{F} = (x - y)\mathbf{i} - y\mathbf{j} + \mathbf{k}$ (b) div $\mathbf{F} = z - \dfrac{1}{2\sqrt{z}}$

7. $\mathbf{F}$ is not conservative

8. $\frac{\pi}{60}(391\sqrt{17} + 1)$