Section 9.1 Three Dimensional Coordinate Systems

<u>Recall</u>: In 2-D we represent points on a plane with ordered pairs.
Coordinate Axes:

The xy-plane contains the x- and y- planes ($z = 0$)
The yz-plane contains the y- and z- planes ($x = 0$)
The xz-plane contains the x- and z- planes ($y = 0$)

Definition These 3 coordinates divide the planes into eight parts, called **octants**. The first octant is determined by the positive axes ($x, y, z > 0$).

Definition We can represent any point in space by an **ordered triple** (a, b, c) of real numbers where a, b, c are known as **coordinates**.

To plot/locate the point (a, b, c) start at the origin, go a units in the x direction, b units in the y direction and c units in the z direction.

Example
Sketch the points (0,5,2), (4,0,-1), (2,4,6) and (1,-1,2) on a single set of coordinate axes

Definition: Given point $P(a, b, c)$, if we drop a perpendicular from P to the xy-plane we get point Q with coordinates $(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, there are projections onto the other two planes, thus creating a rectangular box.

Definition: The **Cartesian product** $\Re \times \Re \times \Re = \{(x, y, z)\,|x, y, z \in \Re\}$ is the set of all ordered triples of real numbers and is denoted by $\Re^3$, **a three-dimensional rectangular coordinate system**.

<u>Note</u>: In 2-D graphs involve x and y so we get lines and curve. In 3-D graphs involve x, y, z so we get surfaces (the simplest of which is a plane).

Example
Describe and sketch the surface in $\Re^3$ represented by the equation $y = x$.

Describe and sketch the surface in $\Re^3$ represented by the equation $x + y = 2$.

Distance Formula in Three Dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example
Find the lengths of the sides of the triangle PQR where $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$. Is it a right triangle? Is it an isosceles triangle?

The Equation of a Sphere

An equation of a sphere with center $C(h, k, l)$ and radius r is $(x-h)^2+(y-k)^2+(z-l)^2 = r^2$. If the center is origin O then the equation of the sphere is $x^2 + y^2 + z^2 = r^2$.

Examples
Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.

Section 9.2 Vectors

Definition: A **vector** is a quantity that has both magnitude and direction; represented by an arrow where the length of the arrow is the magnitude and the arrow points in the direction; denoted: ($\mathbf{v}$) or ($\vec{v}$)

Suppose a particle moves along a line segment from point A to point B:
Definition: The **displacement vector v** has **initial point** A (the tail) and **terminal point** B (the tip); denoted $\mathbf{v} = \mathbf{AB}$.

Combining Vectors

<u>Definition of Vector Addition</u>
If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u** then the **sum u+v** is the vector from the initial point of **u** to the terminal point of **v** (sometimes known as the Triangle Law).

<u>Definition of Scalar Multiplication</u>
If c is a scalar and **v** is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Definition: Two nonzero vectors are **parallel** if they are scalar multiples of one another.
Definition: The **negative** of a vector has the same length but points in the opposite direction.
Definition: The **difference u-v** of given by **u - v = u + (-v)**.

Components

Definition: If we place the initial point of a vector **a** at the origin of a coordinate system, then the coordinates of the terminal point of **a** is known as the components of **a**. (Denoted $\mathbf{a} = \langle a_1, a_2 \rangle$ or $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$).

Given the point $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation **AB** is $\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Example
Find a vector with representation given by the directed line segment **AB** where $A(2, 3)$ and $B(-2, 1)$. Draw **AB** and the equivalent representation starting at the origin.

Definition: The magnitude or length of the vector $\mathbf{v}$ is the length of any of its representation and is denoted by the symbol $|\mathbf{v}|$ or $||\mathbf{v}||$.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$.
The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

If $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and c is a scalar, then $\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2 \rangle$ and $c\mathbf{a} = \langle ca_1, ca_2 \rangle$. This is also similar for 3D vectors.

Example
Find the sum of $\langle 3, -1 \rangle$ and $\langle -2, 4 \rangle$. Calculate the magnitude of this new vector.

Properties of Vectors
If **a**, **b**, **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (\mathbf{-a}) = \mathbf{0}$
5. $c\,(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\,\mathbf{b}$
6. $(c + d)\,\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Definition: Let $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ then any vector in V_3 can be expressed in terms of the **standard basis vectors i, j, and k**.

Examples
Write $\langle 3, 1, -5 \rangle$ using its standard basis.

Find $\mathbf{a} + \mathbf{b}$, $\mathbf{2a} + \mathbf{3b}$,$|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$ where $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

Definition: A **unit vector** is a vector whose length is 1. For instance **i**, **j**, **k** are all unit vectors. In general, the unit vector that has the same direction as **a** is $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ (also called normalizing the vector).

Example
Find a unit vector with the same direction as $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

Applications

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction.

Definition: The **resultant force** is the sum of all the forces acting on an object.

Example
Two forces F_1 and F_2 with magnitudes 10lb and 12lb act on an object at a point P. Find the resultant force $\mathbf{F}$ acting at P as well as its magnitude and its direction.

A clothesline is tied between two poles, 8m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

Section 9.3 The Dot Product

Goal: multiply two vectors together

Work and the Dot Product

Suppose you wish to find the work W done in moving a particle from one point P to another point Q.

Recall: the work done by a constant force F on an object a distance of d units is given by $W = Fd$ (but only applies when the force is in the direction the particle moves).

Suppose this is not the case and the force acts in a different direction.
Definition: Then the **displacement vector** is $\mathbf{D} = \mathbf{PQ}$.

The work done by $\mathbf{F}$ is defined as the magnitude of the displacement multiplied by the magnitude of the applied force in the direction of the motion.

Therefore $|\mathbf{PS}| = |\mathbf{F}| \cos\theta$
So the work done by $\mathbf{F}$: $W = |\mathbf{F}||\mathbf{D}| \cos\theta$

Definition: The **dot product** (or scalar product) of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $0 \leq \theta \leq \pi$. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, we define $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$.

Definition: Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called **perpendicular**, or **orthogonal** if and only if $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$

Examples
Find $\mathbf{a} \cdot \mathbf{b}$ where $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$, and $\theta = 2\pi 3$

A woman exerts a horizontal force of 25 lbs on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20° above the horizontal. Find the work done on the box?

The Dot Product in Component Form

Given two vectors in component form, then by applying the Law of Cosines the dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Properties of the Dot Product

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot \mathbf{b{+}c} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot c\mathbf{b}$
5. $0 \cdot \mathbf{a} = 0$

Example
Find $\mathbf{a} \cdot \mathbf{b}$ where $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 9\mathbf{k}$

Find the angle between the vectors $\mathbf{a} = \langle 0, 1, 1 \rangle$ and $\mathbf{b} = \langle 1, 2, -3 \rangle$.

Projections

Suppose you have vectors $\mathbf{a} = \mathbf{PQ}$ and $\mathbf{b} = \mathbf{PR}$. If S is the "foot" perpendicular from R to the line containing $\mathbf{PQ}$ then the vector with representation $\mathbf{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $proj_a b$.

Definition: The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ is the length of the vector projection, which is the number $|b| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{comp}_a b = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_a b = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

Example
Find the scalar and vector projection of $\mathbf{b} = \langle 1, 2, 3 \rangle$ onto $\mathbf{a} = \langle 3, 6, -2 \rangle$

Section 9.4 The Cross Product

Torque and Cross Product

Definition: If $\mathbf{a}$ and $\mathbf{b}$ are nonzero three-dimensional vectors, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\mathbf{n}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the **right-hand rule**.

The magnitude of torque is the distance from the axis where the force is applied $|\mathbf{r}|$ and the scalar component of the force $\mathbf{F}$ in the direction perpendicular to $\mathbf{r}$.

Example
A bicycle pedal is pushed by a foot with a 60-N force. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.

Facts:

- $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
- Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$
- The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Properties of the Cross Product If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

The Cross Product in Component Form

Suppose $\mathbf{a}$ and $\mathbf{b}$ are given in component form $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$. (an easy way to remember is to think of determinants of 3 by 3 matrices)

Example
Find the cross product of $\mathbf{a} = \langle 1, 2, 0 \rangle$ and $\mathbf{b} = \langle 0, 3, 1 \rangle$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Find two unit vectors orthogonal to both $\langle 2, 0, -3 \rangle$ and $\langle -1, 4, 2 \rangle$

Triple Products

Definition: The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product**.

Facts:

- The volume of the parallelepiped determined by the vectors **a, b, c** is the magnitude of their scalar triple product: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Example
Find the volume of the parallelepiped with adjacent edges PQ, PR, PS where $P(2, 0, -1)$, $Q(4, 1, 0)$, $R(3, -1, 1)$, $S(2, -2, 2)$.

Section 9.5 Equations of Lines and Planes

To form a line, L, in 3D we need a point $P_0(x_0, y_0, z_0)$ on L and the direction of L (which can be described by a vector).

Given a line L, draw a vector $\mathbf{v}$ that is parallel to L.
Let $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ be a point on L where
$\mathbf{r}_0$ and $\mathbf{r}$ are the position vectors for P_0 and P.
If $\mathbf{a}$ is the vector from P_0 to P then the Triangle Law
gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.
Since $\mathbf{a}$ and $\mathbf{v}$ are parallel then there exists a scalar t
such that $\mathbf{a} = t\mathbf{v}$.
Therefore, $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, which is a **vector equation**.

Note: As the **parameter** t varies, the line is traced out by the tip of the vector $\mathbf{r}$. If $t > 0$ then the line is traced to one side of P_0. If $t < 0$ the line goes the other way.

If $\mathbf{v} = \langle a, b, c\rangle$ then $t\mathbf{v} = \langle ta, tb, tc\rangle$. Let $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ then by the vector equation we have $\langle x, y, z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$.

Definition: The **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ are $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$.

Example
Find a vector equation and parametric equations for the line through the point (-2,4,10) and parallel to the vector $\langle 3, 1, -8\rangle$.

Definition: In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L then the numbers a, b, c are called **direction numbers** of L.

Definition: By eliminating the parameter t from the parametric equations we get the **symmetric equations** $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

Note: If one of a, b, c is 0, t can still be eliminated. (e.g. if $a = 0$ then $x = x_0$ which means that L lies in the vertical plane $x = x_0$)

Fact: The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$ where $0 \leq t \leq 1$.

Examples

Find parametric equations and symmetric equations for the line through (1,-1,1) and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$.

Find symmetric equations for the line that passes through the point (0,2,-1) and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$.

Planes

A pair of lines in 3D (or 2D) can either have 0, 1, or infinitely many points of intersection. A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ on the plane and a vector perpendicular to the plane (known as a **normal vector**). If a plane passes through point P and has normal vector $\mathbf{n} =< a, b, c >$ then for any point (x, y, z) in the plane, the vector $< x - x_0, y - y_0, z - z_0 >$ is parallel to the plane.

Since $\mathbf{n}$ is perpendicular to the plane and $< x - x_0, y - y_0, z - z_0 >$ is parallel to the plane, then we know that $\mathbf{n} =< a, b, c >$ is perpendicular to $< x - x_0, y - y_0, z - z_0 >$.
By rules of dot product, that leads us to **the equation of a plane**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which can be simplified to $ax + by + cz = d$.

If given a point and a perpendicular vector it is fairly straightforward to find the equation of a plane.

Example

Find an equation of the plane through the point (6,3,2) and perpendicular to the vector $< -2, 1, 5 >$.

If we know 3 points on the plane, we can also find the equation of a plane.

Example

Find an equation for the plane passing through the points $P(0, 1, 1)$, $Q(1, 0, 1)$ and $R(1, 1, 0)$.

Note: We already have a point (actually 3 points) on the plane, so we need to find a normal vector. To do this we need to take the cross product of two of the vectors in the plane. (Remember to form vectors with the given points and note that these vectors are parallel to the plane).

We now have a normal vector to the plane. Pick one of the points and find the equation of a plane.

Facts:

1. Two vectors are parallel if their normal vectors are parallel.
2. If two vectors planes are not parallel, then they intersect in a straight line.
   Note:The line of intersection is perpendicular to the normal vectors of both planes (because the line is in each plane), so it can be computed using the cross product.

Example

Find an equation of the line that passes through the point (-1,2,1) and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

Section 9.6 Functions and Surfaces

Functions of Two Variables

Definition: A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is $\{f(x, y) | (x, y) \in D\}$.

Definition: The variables x and y are independent variables and $z = f(x, y)$ is the dependent variable.

Examples

Let $f(x, y) = x^2 e^{3xy}$. Find $f(2, 0)$. Find the domain and range of f.

Find and sketch the domain of $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$.

Graphs

Definition: If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\Re^3$ such that $z = f(x, y)$ and (x, y) is in D.

Example

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Definition: A function of the form $f(x, y) = ax + by + c$ is called a **linear function**.

To sketch a graph of two functions:

- determine the shapes of cross-sections (slices) of the graph
- fix x and let y vary so we have a function of one variable $z = f(k, y)$

  - this graph is the curve that results when we intersect the surface $z = f(x, y)$ with the vertical plane $x = k$

- fix y and let x vary so we have a function of one variable $z = f(x, k)$
  - this graph is the curve that results when we intersect the surface $z = f(x, y)$ with the horizontal plane $y = k$

- could also slice with horizontal planes

- these are known as **traces** (or cross sections) of the surface

Quadric Surface
Definition: The graph of a 2nd degree equation in three variables x, y, z is called a **quadric surface**.

Refer to p682 that has graphs of quadratic surfaces.