Section 10.1 Vector Functions and Space Curves

Definition: A **vector function** is a function whose domain is a set of real numbers and whose range is a set of vectors.

Definition: For every number t in the domain of $\mathbf{r}$ there is a unique vector in V_3, denoted by $\mathbf{r}(t)$. If $f(t), g(t), h(t)$ are the components of the vector $\mathbf{r}(t)$ then f, g, h are real-valued functions called the **component functions** of $\mathbf{r}$ (denoted: $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.

Example
Find the domain of the vector function $\mathbf{r}(t) = \left\langle t^2, \sqrt{t-1}, \sqrt{5-t}\right\rangle$.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\lim\limits_{t\to a}\mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\right\rangle$ provided the limits exist.

Example
Find $\lim\limits_{t\to 0^+} \langle \cos t, \sin t, t\ln t\rangle$.

Suppose that f, g, h are real-valued functions on an interval I.
Definition: The set C of all points (x, y, z) in space, where $x = f(t)$, $y = g(t)$, $z = h(t)$ is known as **space curves** where x,y,z are known as **parametric equations of C** and t is called a **parameter**.

Example
Sketch $\mathbf{r}(t) = t^2\mathbf{i} + t^4\mathbf{j} + t^6\mathbf{k}$.

Fact: The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the **vector equation** $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \le t \le 1$.

Examples

Find a vector equation and parametric equations for the line segment that joins $P(0,0,0)$ to $Q(1,2,3)$.

Find a vector function that represents the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

Section 10.2 Derivatives and Integrals of Vector Functions

Derivatives

Definition: The **derivative** $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists. The vector $\mathbf{r}'(t)$ is also called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P.

Definition: The **unit tangent vector** is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Theorem: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, h are differentiable functions then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Differentiation Rules:
Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Example
Find $\mathbf{r}'(t)$ when $\mathbf{r}(t) = \left\langle t-2, t^2+1 \right\rangle$.

Find the derivative of $\mathbf{r}(t) = e^{t^2}\mathbf{i} - \mathbf{j} + \ln(1+3t)\mathbf{k}$.

Find the unit tangent vector $\mathbf{T}(t)$ to $\mathbf{r}(t) = \cos t\ \mathbf{i} + 3t\ \mathbf{j} + 2\sin 2t\ \mathbf{k}$ at the parameter $t = 0$.

Find parametric equations for the tangent line to the curve $x = t^5$, $y = t^4$, $z = t^3$ at the point $(1, 1, 1)$.

The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection.

Integrals

Definition: The **definite integral** of a continuous vector function $\mathbf{r}(t)$ is defined as $\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\ \mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$

Examples

Evaluate $\int_0^1 \left(16t^3\mathbf{i} - 9t^2\mathbf{j} + 25t^4\mathbf{k}\right)\ dt$

Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\ \mathbf{i} + 3t^2\ \mathbf{j} + \sqrt{t}\ \mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

Section 10.3 Arc Length and Curvature

Arc Length

Recall: using parametric equations $x = f(t)$ and $y = g(t)$ the arc length is given by $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

The length of a space curve is defined in exactly the same way

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b |\mathbf{r}'(t)|\, dt$$

Example

Find the length of the curve $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$ where $-10 \le t \le 10$

A single curve C can be represented by more than one vector function. For example, the twisted cube $\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle$, $1 \le t \le 2$ could also be represented by the function $\mathbf{r}_2(t) = \langle e^u, e^{2u}, e^{3u}\rangle$, $0 \le u \le \ln 2$ where the connection between the parameters t and u is given by $t = e^u$ (known as **parameterizations** of curve C)

Definition: Let C be a piecewise-smooth curve given by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$ and C is traversed exactly once as t increases from a to b then the **arc length function** s is defined by $s(t) = \int_a^b |\mathbf{r}'(u)| du = \int_a^b \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$

Example

Reparameterize the curve $\mathbf{r}(t) = 2t\mathbf{i} + (1-3t)\mathbf{j} + (5+4t)\mathbf{k}$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

Curvature

Definition: The **curvature** of a curve at a given point is a measure of how quickly the curve changes direction at that point and is given by $\kappa = \left|\frac{dT}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

In terms of the parameter t it can be defined as $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

Note: The unit normal vector is given by $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

Example
Let $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$. Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Calculate the curvature.

Theorem: The curvature of the curve given by the vector function $\mathbf{r}(t)$ is $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

Example
Find the curvature of $\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle$ at the point (1,1,1).

At what point does the curve $y = e^x$ have maximum curvature? What happens to the curvature as $x \to \infty$?

Section 10.4 Motion in Space: Velocity and Acceleration

Recall: An object's position at any time t is given by $s(t)$. The velocity of this object is given by $v(t) = s'(t)$ and its acceleration is given by $a(t) = s''(t)$.

Suppose that a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. *Definition*: The **velocity vector**, given by $\mathbf{v}(t) = \mathbf{r}'(t)$, and therefore the velocity vector is also the tangent vector and points in the direction of the tangent line.

Definition: The **speed** of the particle at time t is the magnitude of the velocity vector, $|\mathbf{v}(t)|$.

Definition: The **acceleration** of the particle is defined as the derivative of the velocity, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

Example

Find the velocity, acceleration, and speed of a particle with the position function $\mathbf{r}(t) = \langle t^2 + 1, t^3, t^2 - 1 \rangle$.

Find the velocity and position vectors of a particle that has the acceleration function $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j}$ and the given initial velocity $\mathbf{v}(0) = \mathbf{k}$ and position $\mathbf{r}(0) = \mathbf{i}$.

Newton's Second Law of Motion

The vector version of this law states $\mathbf{F}(t) = m\mathbf{a}(t)$.

Example
A force with magnitude 20N acts directly upward from the xy-plane on an object with mass 4kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find its position function and its speed at time t.

Suppose a projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. If air resistance is negligible and the only external force is die to gravity the parametric equations of the trajectory are therefore $x = (v_0 \cos\alpha)t$, $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$.

Example
A projectile is fired with an initial speed of 500 m/s and angle of elevation 30°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.