13.1 Vector Fields

Definition: Let D be a set in $\Re^2$. A **vector field on** $\Re^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

To picture a vector field, draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Since $\mathbf{F}(x, y)$ is a 2-D vector, we can write it in terms of its **component functions** P and Q

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

Note: P and Q are sometimes called **scalar functions**

The same idea of vector fields in 2D can apply in 3D.

Examples

Sketch the vector field $\mathbf{F}(x, y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$.

Sketch the vector field $\mathbf{F}(x, y) = y\mathbf{i} + \frac{1}{2}\mathbf{j}$.

Note: Most vector fields are impossible to draw in 3D.

Gradient Fields

Recall: The gradient of f is defined by $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$.

Therefore, ∇f is really a vector field and is called a **gradient vector field**.

Example

Find the gradient vector field of $f(x, y) = \ln(x + 2y)$.

Find the gradient vector field of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

Find and sketch the gradient vector field of $f(x, y) = xy - 2x$.

13.2 Line Integrals

Goal: Instead of integrating over an interval $[a, b]$ we integrate over a curve C. (known as **line integrals**)

Let C be a smooth curve defined by the parametric
equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
Therefore C has vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.
Divide the parameter interval $[a, b]$ into n subintervals
of equal width. Let $x_i = x(t_i)$ and $y_i = y(t_i)$.
The corresponding points $P_i(x_i, y_i)$ divide C into n
subarcs with lengths Δs_1, Δs_2, Δs_3...
Pick a point in the i^{th} subarc $P_i^*(x_i^*, y_i^*)$.

We can now form the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$ where f is a function of two variables whose domain includes C.

Definition: If f is defined on C then the line integral of f along C is $\displaystyle\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\delta s_i$.

Recall: Arc Length: $\displaystyle L \approx \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

Therefore, $\displaystyle\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

Examples

Evaluate $\displaystyle\int_C y ds$ where $C : x = t^2, y = t, 0 \leq t \leq 2$.

Evaluate $\displaystyle\int_C xy^4 ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

Note: If C is a piecewise-smooth curve then $\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \ldots + \int_{C_n} f(x,y)ds$

If we replace Δs_i with either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ then we have the **line integrals of f along C with respect to x and y.**

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

Note: These line integrals can occur together $\int_C P(x,y)dx + \int_C Q(x,y)dy$.

Note: It is sometimes hard to think of parametric equations for curves.

Recall: vector representation of a line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, $0 \leq t \leq 1$.

Example

Evaluate $\int_C xe^{yz}$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

Note: In general, the value of a line integral depends not just on the endpoints of the curve, but also on the path. If C_1 denotes the path from A to B and $\int_a^b f(x,y)ds = L$ then $-C_1$ is the the path from B to A and $\int_a^b f(x,y)ds = L$.

Definition: A given parameterization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ determines an **orientation**, or **direction** of a curve C, with the positive direction corresponding to increasing values of t.

Line Integrals in Space

Let C be defined by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.
If f is a function of 3 variables then

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

If $f(x, y, z) = 1$ then $\int_C ds = \int_a^b |\mathbf{r}'(t)| dt = L$ where L is the length of the curve C.

Example

Evaluate $\displaystyle\int_C (2x+9z)ds$ where $C : x = t, y = t^2, z = t^3$, $0 \le t \le 1$.

Line Integrals of Vector Fields

Recall: $W = \displaystyle\int_a^b f(x)dx$ (Section 6.5) and $W = \mathbf{F}\cdot\mathbf{D}$ where D is the displacement vector (Section 9.3)

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\Re^3$ (gravitational field).

Goal: Compute the work done by this force in moving a particle along a smooth curve C.

Start as before: Divide C into subarcs with lengths δs_i. Choose a point P_i^* on the i^{th} subarc. If δs_i is small then as the particle moves from P_{i-1} to P_i it proceeds in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector.

Definition: The **work** W done by the force field $\mathbf{F}$ is $W = \displaystyle\int_C \mathbf{F}(x,y,z)\cdot\mathbf{T}(x,y,z)ds$. Also,

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|dt = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt$$

So, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_C \mathbf{F}\cdot\mathbf{T}ds$ is the line integral of $\mathbf{F}$ along C.

Example

Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and C is given by the vector equation $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

13.3 The Fundamental Theorem for Line Integrals

Definition: A vector field is called a **conservative vector field** if it is the gradient of some scalar function. i.e There exists a function f such that $\mathbf{F} = \nabla f$.

Recall: Part 2 of the Fundamental Theorem of Calculus says

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$.

Let the gradient vector, ∇f, of a function f of 2 or 3 variables be a "derivative" of f, then

Theorem: The Fundamental Theorem for Line Integrals
Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \leq t \leq b$. Let f be a differentiable function of 2 or 3 variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note:

- We can evaluate the line integral of a conservative vector field, or gradient vector field, simply by knowing the value of f at the endpoints of C.
- If f is a function of 2 or 3 variables with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$ then
  - $\displaystyle\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$
  - $\displaystyle\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$

Independence of Path
Suppose C_1 and C_2 are two piecewise-smooth curves (called **paths**) that have the same initial point A and terminal point B.
In general, $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. From the fundamental theorem of line integrals

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

The line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

Definition: If $\mathbf{F}$ is a continuous vector field then the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent**

of path if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for 2 curves with the same initial and terminal points.
Definition: A curve is called **closed** if its terminal point coincides with its initial point $\mathbf{r}(b) = \mathbf{r}(a)$.
Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D (domain) if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.
Definition: D is **open** if for every point P in D there is a disk with center P that lies entirely in D.
Definition: D is **connected** if any two points in D can be joined by a path that lies in D.

Theorem: Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$. i.e. Only vector fields independent of paths are conservative.

Is it possible to determine whether or not a vector field $\mathbf{F}$ is conservative?
Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives.Then

$$P = \frac{\partial f}{\partial x} \text{ and } Q = \frac{\partial f}{\partial y}$$

By Clairaut's Theorem $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x}$.
Theorem: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ iff $\mathbf{F}$ is a conservative vector field.

Examples
Determine whether or not $\mathbf{F}(x, y) = xe^y\mathbf{i} + ye^x\mathbf{j}$ is a conservative vector field.

Determine whether or not $\mathbf{F}(x, y) = (2x\cos y - y\cos x)\mathbf{i} + (-x^2\sin y - \sin x)$ is a conservative vector field.

Find a function f such that $\mathbf{F} = \nabla f$ and use f to evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

(a) $\mathbf{F}(x, y) = x^3y^4\mathbf{i} + x^4y^3\mathbf{j},\ C : \mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 + t^3)\mathbf{j}, 0 \leq t \leq 1$

(b) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

(c) $\mathbf{F}(x, y, z) = y^2 \cos z\mathbf{i} + 2xy \cos z\mathbf{j} - (xy^2 \sin z\mathbf{k}$ $C : \mathbf{r}(t) = t^2\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 1.$

13.4 Green's Theorem

Gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C

Definition: The **positive orientation** of a simple closed curve C refers to a single *counter-clockwise* traversal of C.

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA \right)$$

Other notation: $\oint_C P\,dx + Q\,dy$ or $\int_{\partial D} P\,dx + Q\,dy$

Note: Green's Theorem is the counterpart of the Fundamental Theorem of Calculus for double integrals.

Examples

Evaluate $\oint xy^2\,dx + x^3\,dy$, C is the rectangle with vertices $(0,0), (2,0), (2,3)$, and $(0,3)$ directly and then by using Green's Theorem.

The reverse direction of Green's Theorem is if

- it is easier to compute the line integral
- computing areas

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Example

Use Green's Theorem to evaluate $\int_C e^y dx + 2xe^y dy$ where C is the square with sides $x = 0, x = 1, y = 0, y = 1$.

13.5 Curl and Divergence

Curl

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field.
Defintion: The **curl** of $\mathbf{F}$ is defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Let ∇, "del", be the vector differential operator i.e.

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

If we think of ∇ as a vector with components, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\
&= \text{curl } \mathbf{F}
\end{aligned}$$

Therefore, the easiest way to remember the definition of curl is

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F}$$

Example
Find the curl of the vector field $\mathbf{F}(x, y, z) = xyz\ \mathbf{i} - x^2y\ \mathbf{k}$

Theorem: If f is a function of 3 variables that has continuous second order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, then the theorem can be rephrased as: If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$

Theorem If $\mathbf{F}$ is a vector field whose component functions have continuous partial derivatives and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

Example

Determine whether or not $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$ is conservative. If it is conservative find a function f such that $\mathbf{F} = \nabla f$.

Divergence

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field.

Definition: The **divergence of F** is defined by

$$\mathrm{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Note: curl $\mathbf{F}$ is a vector field and div $\mathbf{F}$ is a scalar field.
This can be rewritten as

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

Example

Find the divergence of $\mathbf{F}(x, y, z) = xyz\ \mathbf{i} - x^2y\ \mathbf{k}$

Theorem: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field with P, Q, R having continuous second-order partial derivatives, then div curl $\mathbf{F}$=$\mathbf{0}$.

Vector Forms of Green's Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\mathrm{curl}\mathbf{F}) \cdot \mathbf{k} dA$$

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \mathrm{div}\mathbf{F}(x, y) dA$$

10.5 Parametric Surfaces

Recall: A space curve can be described by a vector function $\mathbf{r}(t)$.
We can describe a surface by a vector function $\mathbf{r}(u,v)$ of 2 parameters.
Let,

$$\mathbf{r}(u,v) = x(u,v)\ \mathbf{i} + y(u,v)\ \mathbf{j} + z(u,v)\ \mathbf{k}$$

be a vector-valued function defined on a region D in the $uv-$ plane.
Defintion: The set of points (x,y,z) such that

$$x = x(u,v) \qquad y = y(u,v), \qquad z = z(u,v)$$

and (u,v) varies throughout D is called a **parametric surface** S and the equations for x, y, z are **parametric equations**

Example
Identify the surface with vector equation $\mathbf{r}(u,v) = (u+v)\mathbf{i} + (3-v)\mathbf{j} + (1+4u+5v)\mathbf{k}$

If a parametric surface S is given by a vector function $\mathbf{r}(u,v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These curves are called **grid curves**.

Examples
Find a parametric representation for the plane that passes through the point (1,2,-3) and contains the vectors $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$.

Find a parametric representation for the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ that lies to the right of the xz-plane.

Find a parametric representation for the part of the cylinder $y^2 + z^2 = 16$ that lies between the planes $x = 0$ and $x = 5$.

Tangent Planes to Parametric Surfaces

Goal: Find the tangent plane to a parametric surface S traced out by a vector function $\mathbf{r}(u,v) = x(u,v)\ \mathbf{i} + y(u,v)\ \mathbf{j} + z(u,v)\ \mathbf{k}$ at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

The tangent vector to a curve C_1 lying on S (keep u constant by $u = u_0$) is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 that lies on S and its tangent vector at P_0 is

$$\mathbf{r}_v = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

Note:

- If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface is called **smooth** (it has no "corners").
- The **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Example
Find an equation of the tangent plane to the parametric surface $x = u+v,\ y = 3u^2,\ z = u-v$ at the point $(2, 3, 0)$.

12.6 Surface Area

Goal: apply double integrals to the problem of computing the area of a surface

Recall: a parametric surface S is defined by a vector-valued function of two parameters $textbfr(u,v) = x(u,v)\ \mathbf{i} + y(u,v)\ \mathbf{j} + z(u,v)\ \mathbf{k}$ or by parametric equations $x = x(u,v), y = y(u,v), z = z(u,v)$.

To find the area of S, divide S into patches and approximate the area of each patch by the area of a piece of a tangent plane.

Recall: From 11.4, the tangent vector to C_1 $(u = u_0)$at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

Similarly, the tangent vector to C_2 is given by

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

Now we will define the surface area of a general parametric surface.

Consider a surface whose parameter domain D is a rectangle and divide it into subrectangles R_{ij}. Choose $\left(u_i^*, v_j^*\right)$ be the lower left corner of R_{ij}. The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}\left(u_i^*, v_j^*\right)$ as one of its corners.

Let $\mathbf{r}_u^* = \mathbf{r}_u\left(u_i^*, v_j^*\right)$ and $\mathbf{r}_v^* = \mathbf{r}_v\left(u_i^*, v_j^*\right)$ be the tangent vectors at P_{ij}.

The two edges of the patch that meet at P_{ij} can be approximated using vectors. These vectors can be approximated by the vectors $\Delta u\mathbf{r}_u^*$ and $\Delta v\mathbf{r}_v^*$. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\mathbf{r}_u^*$ and $\Delta v\mathbf{r}_v^*$. The area of this parallelogram is

$$\left|(\Delta u\mathbf{r}_u^*) \times (\Delta v\mathbf{r}_v^*)\right| = \left|\mathbf{r}_u^* \times \mathbf{r}_v^*\right| \Delta u \Delta v$$

Definition: If a smooth parametric surface S is given by the equation $textbfr(u,v) = x(u,v)\ \mathbf{i} + y(u,v)\ \mathbf{j} + z(u,v)\ \mathbf{k}$ and S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D \left|\mathbf{r}_u \times \mathbf{r}_v\right| dA$$

where $\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$

Example
Find the area of the part of the plane $z = 2+3x+4y$ that lies above the rectangle $[0,5]\times[1,4]$

Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2$.

Surface Area of a Graph

For the special case of a surface S with equation $z = f(x,y)$ where (x,y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \; y = y \; z = f(x,y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \; \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Therefore

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example
Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

13.6 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.

Parametric Surfaces

Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D$$

Assume that D is a rectangle and divide it into subrectangles R_{ij} with dimensions Δu and Δv. Then S is divided into patches S_{ij}. Evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} and form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

Definition: The **surface integral of f over the surface S** is defined as $\displaystyle\iint_S f(x,y,z)dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$

Recall: In Section 12.6, $\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\Delta v$.

Therefore, $\displaystyle\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|dA$.

Example

Evaluate $\displaystyle\iint_S yz\,dS$ where S is the surface with parametric equations $x = u^2$, $y = u\sin v$, $z = u\cos v$, $0 \le u \le 1$, $0 \le v \le \pi/2$.

Applications

Definition: The total **mass** of the sheet is $m = \displaystyle\iint_S \rho(x,y,z)dS$.

Definition: The **center of mass** is $(\bar{x},\bar{y},\bar{z})$ where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)dS, \;\; \bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)dS \;\; \bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)dS$$

Graphs

Recall: Any surface with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x, \;\; y = y, \;\; z = g(x, y)$$

where

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore,

$$\iint_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane.

Example

Evaluate $\iint_S x^2yz\, dS$ where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$.

Evaluate $\iint_S yz\, dS$ where S is the part of the plane $x+y+z = 1$ that lies in the first octant.

Evaluate $\iint_S y\, dS$ where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$.

Evaluate $\iint_S (x^2 y + z^2)\, dS$ where S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$.

Oriented Surfaces

For a **closed surface**, that is, a surface that is the boundary of solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation.

Surface Integrals of Vector Fields

Definition: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

This integral is also called the **flux of F** across S.

If S is given by a vector function $\mathbf{r}(u,v)$ and $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

Examples

Find the flux of $\mathbf{F}$ across S.

(a) $\mathbf{F}(x,y,z) = xy\ \mathbf{i} + yz\ \mathbf{j} + zx\ \mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, and has upward orientation.

(b) $\mathbf{F}(x,y,z) = xze^y\ \mathbf{i} - xze^y\ \mathbf{j} + z\ \mathbf{k}$, S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation.

(c) $\mathbf{F}(x,y,z) = y\ \mathbf{j} - z\ \mathbf{k}$, S consists of the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and the disk $x^2 + z^2 \leq 1$, $y = 1$

An application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$.

Definition: The **heat flow** is defined as the vector field $\mathbf{F} = -K\nabla u$ where K is an experimentally determined constant called the **conductivity**.

The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

Example

The temperature at a point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 9$, $0 \leq x \leq 4$.

13.7 Stokes' Theorem

Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region $\Re^3$ that contains S. Then

$$\iint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_D \mathbf{F} \cdot \mathbf{T} ds \text{ and } \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Example

Use Stokes' Theorem to evaluate $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x^2 e^{yz}\mathbf{i} + y^2 e^{xz}\mathbf{j} + z^2 e^{xy}\mathbf{k}$ where S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$, oriented upward.

Use Stokes Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. C is oriented clockwise as viewed from above.

(a) $\mathbf{F}(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$ C is the triangle with vertices (1,0,0), (0,1,0), and (0,0,1).

(b) $\mathbf{F}(x, y, z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$ C is the circle $x^2 + y^2 = 16, z = 5$

13.8 The Divergence Theorem

The vector version of Green's Theorem is

$$\int_C \mathbf{F}\cdot\mathbf{n}ds = \iint_D \text{div } \mathbf{F}(x,y)dA$$

where C is the positively oriented boundary curve of the plane region D.

Definition: regions E that are simulataneously of types 1, 2, and 3 are called **simple solid regions**.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with the positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}dV$$

The Divergence Theorem states that the flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E.

Example
Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x,y,z) = 3x\mathbf{i} + xy\mathbf{j} + 2xz\mathbf{k}$ on E where E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

(a) $\mathbf{F}(x,y,z) = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j} + yz^2\mathbf{k}$, S is the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$

(b) $\mathbf{F}(x,y,z) = \left(\cos z + xy^2\right)\mathbf{i} + xe^{-z}\mathbf{j} + \left(\sin y + x^2 z\right)\mathbf{k}$ where S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

(c) $\mathbf{F}(x,y,z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$ where S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.