12.1 Double Integrals over Rectangles

Recall: To calculate the area under a curve $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \dfrac{b-a}{n}$ using rectangles.

Similarly, in 3-D instead of finding area under a curve we will now add depth and find volume under a curve.

Consider a function f of two variables defined on a closed rectangle $R = \{(x, y) \in \Re^2 | a \leq x \leq b, c \leq y \leq d\}$. Let S be the solid that lies above R and under the graph of f.

Goal: Find the volume of S.

Divide rectangle R into subrectangles. Divide $[a, b]$ into m subintervals with width $\Delta x = (b - a)/m$ and divide $[c, d]$ into n subintervals of equal width $\Delta y = (c - d)/n$, therefore, forming subrectangles each with area $\Delta A = \Delta x \Delta y$.
If we choose a sample point in (x_{ij}, y_{ij}) in each subrectangle, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box with base R_{ij} and height $f(x_{ij}, y_{ij})$.
If we follow this procedure for all the rectangles and add the volumes we get the approximation to the total volume:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

Definition: The **double integral** of f is given by

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_h)\Delta A$$

if this limit exists.

If $f(x, y) \geq 0$, then the volume V of the solid that lies above rectangle R and below surface $z = f(x, y)$ is $V = \iint_R f(x, y)dA$.

Example
Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R \sin(x+y)dA$ where $R = [0,\pi] \times [0,\pi]$. Take the sample points to be lower left corners.

The Midpoint Rule

$$\iint_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example
Use the midpoint rule to estimate the integral above.

Average Value

The average value of a function f of two variables is

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)dA$$

where $A(R)$ is the area of R.
Properties of Double Integrals

1. $\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$
2. $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$ where c is a constant
3. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$

12.2 Iterated Integrals

Suppose that f is a function of 2 variables.

Notation: $\int_c^d f(x,y)dy$ means that x is held fixed and f is integrated with respect to y from $y = c$ to $y = d$.

Let $A(x) = \int_c^d f(x,y)dy$. If we integrate A with respect to x from $x = a$ to $x = b$ we get $\int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$ known as an **iterated integral**.

Note: Work from the inside out.

Example

$\int_1^3 \int_0^1 (1 + 4xy)\, dx\, dy$

$\int_0^2 \int_0^{\pi/2} x \sin y\, dy\, dx$

Fubini's Theorem

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Example

Calculate $\iint_R \frac{xy^2}{x^2+1}\, dA, R = \{(x,y)|0 \le x \le 1, -3 \le y \le 3\}$

Examples

Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = \{(x, y)|0 \leq x \leq 1, -2 \leq y \leq 3\}$.

Find the volume of the solid bounded by the surface $z = x\sqrt{x^2 + y}$ and the planes $x = 0, x = 1, y = 0, y = 1$, and $z = 0$.

Calculate $\displaystyle\int_0^2 \int_0^{\pi/2} x \sin y \, dy \, dx$.

12.3 Double Integrals over General Regions

We don't have to integrate over rectangles, or within a certain interval. We could integrate a region D that is bounded by enclosing it in a rectangular region.

Type I

Definition: A region D is said to be **type I** if it lies between the graphs of two continuous functions of x. i.e.

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

Then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Example

$\iint_D x^3 y^2\, dA,\ D = \{(x, y) | 0 \leq x \leq 2, -x \leq y \leq x\}$

Type II

Definition: A region D is said to be **type II** if it lies between the graphs of two continuous functions of y. i.e.

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

Then

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example

$\iint_D e^{y^2}\, dA,\ D = \{(x, y), 0 \leq y \leq 1, 0 \leq x \leq y\}$

Example
Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices (1,1), (4,1), (1,2).

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z, x = 0, z = 0$ in the first octant.

Example
Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ by reversing the order of integration.

Evaluate $\iint_D (x^2 \tan x + y^3 + 4) dA$ where $D = \{(x, y) | x^2 + y^2 \leq 2\}$. [Note: D is symmetric with respect to both axes.]

12.4 Double Integrals in Polar Coordinates

Appendix H - Polar Coordinates

A point $P(x, y)$ can be given in terms of polar coordinates (r, θ) where r is the distance from the origin to a point P and θ is the angle that is made.

Example
Convert $(2, \frac{\pi}{3})$ from polar coordinates to Cartesian coordinates.

Represent (1,-1) in polar coordinates.

Double Integrals in Polar Coordinates

- easier to work with when dealing with circular regions
- points described by (r, θ) rather than (x, y) and area of small region not $dx\,dy$ but $r\,dr\,d\theta$

Recall: Polar coordinates (r, θ) of a point are related to rectangular coordinates by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

Consider the polar rectangle $R = \{(r, \theta)\,|a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

In order to compute the double integral $\iint_R f(x, y)dA$, we divide the interval $[a, b]$ into m subintervals of equal width $\Delta r = \dfrac{b-a}{m}$ and the interval $[\alpha, \beta]$ into n subintervals of equal width $\Delta\theta = \dfrac{\beta - \alpha}{n}$. R is now divided into smaller polar rectangles.

The area of one of these polar rectangles is $\Delta A \approx r\Delta r\Delta\theta$ (comes from area of a sector) which implies that $dA = r\,drd\theta$

Then the double integral over the polar rectangle R is

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\,drd\theta$$

<u>Note</u>:

- convert from rectangular to polar coordinates

  replace: $x = r\cos\theta$ and $y = r\sin\theta$

  change limits of integration

  replace dA with $r\,drd\theta$

- DO NOT forget the r in the $r\,drd\theta$

Examples

Evaluate $\iint_R \cos(x^2+y^2)\,dA$ where R is the region that lies above the $x-$axis within the circle $x^2+y^2=9$.

Evaluate $\iint_R \sqrt{4-x^2-y^2}\,dA$, where $R = \{(x,y)|x^2+y^2 \leq 4, x \geq 0\}$

This concept can be extended to more complicated regions (similar to Type II regions) i.e. $D = \{(r,\theta)\,|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ Therefore,

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\,drd\theta$$

Examples

Find the volume under the cone $z = \sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \leq 4$.

Find the volume inside both the cylinder $x^2+y^2 = 4$ and the ellipsoid $4x^2+4y^2+z^2 = 64$.

Examples

Evaluate $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

Evaluate $\displaystyle\int_{0}^{1}\int_{y}^{\sqrt{2-y^2}} x+y\,dx\,dy$ by converting to polar coordinates.

12.5 Application of Double Integrals

We can use double integrals to

- computing volumes
- finding areas of surfaces (Section 12.6)
- computing mass, electric charge, center of mass, moments of inertia

Density and Mass

For a lamina (thin plate) with constant density, mass = density $\times$ area.
If the density is not constant, then the total mass is given by

$$m = \iint_D \rho(x, y) dA$$

Note: This idea can be applied to an electric charge distributed over a region D

$$Q = \iint_D \sigma(x, y) dA$$

Example
Electric charge is distributed over the rectangle $1 \leq x \leq 3, 0 \leq y \leq 2$ so that the charge density at (x, y) is $\sigma(x, y) = 2xy + y^2$. Find the total charge on the rectangle.

Moments and Centers of Mass

Consider a lamina with variable density.

Recall: The moment of a particle about an axis is the product of its mass and its directed distance from the axis.
The moment of the entire lamina is

$$M_x = \iint_D y\rho(x, y) dA \text{ about the x-axis}$$

$$M_y = \iint_D x\rho(x, y) dA \text{ about the y-axis}$$

As before, the center of mass is given by $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y) dA \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y) dA \text{ where } m = \iint_D \rho(x, y) dA$$

Examples

Find the mass and center of mass of the lamina that occupies the region where D is bounded by $y = e^x, y = 0, x = 0, x = 1$ and has the density function $\rho(x, y) = y$.

A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

Moment of Inertia

Definition: The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. The moment of inertia of a lamina is given by

$$I_x = \iint_D y^2 \rho(x, y) dA \text{ about the x-axis} \qquad I_y = \iint_D x^2 \rho(x, y) dA \text{ about the y-axis}$$

The moment of inertia about the origin (**polar moment of inertia**) is given by

$$I_0 = \iint (x^2 + y^2) \rho(x, y) dA$$

Example

Find the moments of inertia I_x, I_y, I_0 for the lamina of the above example.

12.7 Triple Integrals

So far we have single integrals for functions of one variable and double integrals for functions of two variables. Therefore we can define triple integrals for functions of three variables.

Let f be the function defined on the rectangular box $B = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$.

<u>Definition</u>: The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV$$

By Fubini's Theorem

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Example

Evaluate the iterated integrals $\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$.

In computing triple integrals over non-rectangular regions, we need to find one variable that can be the direction of our first first integral. If we can successfully compute the innermost integral, then the problem has been reduced from a triple integral to a double integral.

3 Cases

Definition: A solid region is said to be **Type 1** if it lies between the graphs of two continuous functions of x and y

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \Rightarrow$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$

where D is the projection of E onto the xy-plane.

Definition: A solid region is said to be **Type 2** if it lies between the graphs of two continuous functions of y and z

$$E = \{(x, y, z)|(y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\} \Rightarrow$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx\right] dA$$

where D is the projection of E onto the yz-plane.

Definition: A solid region is said to be **Type 3** if it lies between the graphs of two continuous functions of x and z

$$E = \{(x, y, z)|(x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\} \Rightarrow$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA$$

where D is the projection of E onto the xz-plane.

Note: Some regions fit more than one form, but the difficulty of integration may depend on which is chosen.

Examples

Evaluate $\iiint_E 2x\, dV$, where $E = \{(x, y, z)|0 \leq y \leq 2, 0 \leq x \leq \sqrt{4 - y^2}, 0 \leq z \leq y\}$.

Write five other iterated integrals that are equal to $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$.

Applications of Triple Integrals

Recall: The single integral represents the area under the curve. The double integral represents the volume under the surface. The triple integral represents the "hypervolume" of a 4D object.

If $f(x, y, z) = 1$ then the triple integral represents the volume of E:

$$V(E) = \iiint_E dV$$

Example

Evaluate the triple integral $\iiint_E xy\, dV$, where E is the solid tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 2, 0)$, and $(0, 0, 3)$.

Find the volume of the solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$.

The **mass** of an object is $\iiint_E \rho(x, y, z)\, dV$ and its **moments** about the 3 coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)dV \qquad M_{xy} = \iiint_E z\rho(x, y, z)dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

The **moments of inertia** about the the 3 coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV \quad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$$

The total **electric charge** having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

The **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\, dV$$

Examples

Find the mass and center of mass of the solid where E is the cube given by $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$ with the $\rho(x, y, z) = x^2 + y^2 + z^2$.

The joint density function for random variables X, Y, and Z is $f(x, y, z) = Cxyz$ if $0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2$ and $f(x, y, z) = 0$ otherwise.

(a) Find the value of the constant C.

(b) Find $P(X \leq 1, Y \leq 1, Z \leq 1)$.

(c) Find $P(X + Y + Z \leq 1)$.

Section 9.7 Cylindrical and Spherical Coordinates

In 2-D use polar coordinates to describe certain curves/regions.
In 3-D there are 2 coordinate systems that are similar to polar coordinates and give convenient descriptions of some surfaces/solids.

Cylindrical Coordinates

Definition: In the **cylindrical coordinate system** a point P in 3D is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy plane to P

To convert from cylindrical to rectangular coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

To convert from rectangular to cylindrical coordinates

$$r^2 = x^2 + y^2 \quad \tan\theta\frac{y}{x} \quad z = z$$

Example
Change $(1, -1, 4)$ from rectangular to cylindrical coordinates.

Spherical Coordinates

Definition: The **spherical coordinates** (ρ, θ, ϕ) of a point P, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.
Note: $\rho \geq 0 \quad 0 \leq \phi \leq \pi$.

To convert from spherical to rectangular coordinates

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$$

To convert from rectangular to spherical coordinates

$$\rho^2 = x^2 + y^2 + z^2$$

Examples
Change $(1, \sqrt{3}, 2\sqrt{3})$ from rectangular to spherical coordinates.

Write $x^2 + y^2 = z$ in cylindrical coordinates and in spherical coordinates.

Section 12.8.1 Triple Integrals in Cylindrical Coordinates

Recall: The cylindrical coordinates of a point P are (r, θ, z).

Suppose that E is a type 1 region whose projection D on the xy-plane is described in polar coordinates. Let $E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$ and D is given in polar coordinates by $D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$.

The formula for **triple integration in cylindrical coordinates** is

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Examples

Evaluate the integral $\displaystyle\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr$.

Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2 + y^2}\, dV$ where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Section 12.8.2 Triple Integrals in Spherical Coordinates

Recall: The spherical coordinates of a point P are (ρ, θ, ϕ) where the relationship between rectangular and spherical coordinates is

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$$

In this coordinate system the counterpart of a rectangular box is a **spherical wedge** $E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$ By dividing our region into smaller spherical wedges we get

The formula for **triple integration in spherical coordinates** is

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Examples

Evaluate the integral $\displaystyle\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Use spherical coordinates to evaluate $\iiint_E z\,dV$ where E lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.

Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.