Test 2 Review

General Things

- 100 points possible
- NO CALCULATOR of any kind will be needed or allowed
- covers Chapter 11: Sections 11.1 - 11.7
- There are 7 questions (including 1 conceptual question)
- Extra Credit is available and will be due on Tuesday, October 9, 2007

Section 11.1

- Ways to Visualize Functions of Two or Three Variables
  - Contour Map - A contour map consists of level curves of the function which are horizonal traces of the graph of the function projected onto the xy-plane.
  - In 3 variables, examine level surfaces $f(x, y, z) = k$, where k is a constant.

Section 11.2

- $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f.
- To show a limit at a point does not exist, find two different paths approaching the point along which $f(x, y)$ has different limits.
- A function of two variables is called **continuous** at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.
- If f is continuous, the graph will appear as a surface without holes or breaks.

Section 11.3

- Rules to Finding Partial Derivatives of $z = f(x, y)$
  1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
  2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.
- **Clairaut's Theorem** - If the functions f_{xy} and f_{yx} are both continuous, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Section 11.4

- Equation of tangent plane to surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- Linearization or linear approximation of f at (a, b) is

$$L(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Geometrically, the linearization of f at (a, b) is the linear function which approximates f near (a, b).
- If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).
- The **total differential** is defined by $dz = f_x(x, y)dx + f_y(x, y)dy$.

Section 11.5

- **The Chain Rule (Case 1)**: Suppose that $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

- **The Chain Rule (Case 2)**: Suppose that $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$ then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- **Implicit Differentiation**

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Section 11.6

- The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} =< a, b >$ is

$$\begin{aligned} D_u f &= f_x(x, y)a + f_y(x, y)b \\ &=< f_x, f_y > \bullet < a, b > \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$

- The **gradient** of $f(x, y)$ is defined by

$$\nabla f =< f_x, f_y >= f_x\mathbf{i} + f_y\mathbf{j}$$
$$\nabla f =< f_x, f_y, f_z >= f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

- The gradient vector points in the direction of maximum rate of increase of the function.
- On a graph of the function, the gradient points in the direction of steepest ascent.
- The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$.

Section 11.7

- f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.
- If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or $\nabla f(a, b) = 0$.
- **Second Derivatives Test**

  (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

  (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

  (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. (saddle point)

  (d) If $D = 0$ then nothing can be concluded.

  where $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

Practice Problems

1. Find the domain of the function $f(x, y) = \ln(x + y + 1)$.

2. Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4 + y^2}$.

3. Find the first partial derivative of $u = e^{-r}\sin 2\theta$.

4. Find all second partial derivatives of $f(x, y) = 4x^3 - xy^2$.

5. Find an equation of the tangent plane to $z = e^x \cos y$ at $(0, 0, 1)$.

6. Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point (2,3,4) and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

7. If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, find $\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial v}{\partial t}$ when $s = 0$ and $t = 1$

8. Find the directional derivative of $f(x, y, z) = x^2y + x\sqrt{1 + z}$ at the point (1,2,3) in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

9. Find the local max/min, saddle points of $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$

Answers

1. $\{(x, y)|y > -x - 1\}$

2. limit DNE

3. $u_r = -e^{-r}\sin 2\theta$ $u_\theta = 2e^{-r}cos2\theta$

4. $f_x = 12x^2 - y^2, f_y = -2xy, f_{xx} = 24x, f_{yy} = -2x, f_{xy} = f_{yx} = -2y$

5. $z = x + 1$

6. $L(x, y, z) \approx 60x + \dfrac{24}{5}y + \dfrac{32}{5}z - 120;\ \ 38.656$

7. $\dfrac{\partial v}{\partial s} = 5 \quad \dfrac{\partial v}{\partial t} = 0$

8. $D_u f(1, 2, 3) = \dfrac{25}{6}$

9. (-4,1) is a local minimum