Final Exam Practice Problems

Chapter 9

1. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{j} - 5\mathbf{k}$
   (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
   (b) The angle between $\mathbf{a}$ and $\mathbf{b}$.
2. Find two unit vectors that are orthogonal to both $\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
3. Given the points $A(1,0,1)$, $B(2,3,0)$, $C(-1,1,4)$, and $D(0,3,2)$, find the volume of the parallelpiped with adjacent edges AB, AC, and AD.
4. Find a vector perpendicular to the plane through the points $A(1,0,0)$, B(2,0,-1), and $C(1,4,3)$.
5. Find parametric equations for the line through $(1,0,-1)$ and parallel to the line $\frac{1}{3}(x-4) = \frac{1}{2}y = z + 2$.
6. Find an equation of the plane through (3,-1,1), $(4,0,2)$, and $(6,3,1)$.
7. Find an equation of the plane through (1,2,-2) that contains the line $x = 2t$, $y = 3 - t$, and $z = 1 + 3t$.
8. Find the domain $f(x,y) = x\ln(x - y^2)$.
9. Write $x^2 + y^2 + z^2 = 4$ in cylindrical coordinates and in spherical coordinates.

Chapter 10

1. Let $\mathbf{r}(t) = \left\langle \sqrt{2-t}, \frac{(e^t-1)}{t}, \ln(t+1) \right\rangle$.
   (a) Find the domain of $\mathbf{r}$.
   (b) Find $\lim_{t\to 0} \mathbf{r}(t)$.
   (c) Find $\mathbf{r}'(t)$.
2. If $\mathbf{r}(t) = t^2\ \mathbf{i} + t\cos \pi t\ \mathbf{j} + \sin \pi t\ \mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t)dt$.
3. Let C be the curve with equations $x = 2 - t^3$, $y = 2t - 1$, $z = \ln t$.
   (a) Find the point where C intersects the xz-plane.
   (b) Find parametric equations of the tangent line at $(1,1,0)$.
   (c) Find an equation of the normal plane to C at $(1,1,0)$.

4. For the curve given by $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, \frac{1}{2}t^2, t \right\rangle$
   (a) Find the unit tangent vector.
   (b) Find the unit normal vector.
   (c) Find the curvature.
5. A particle moves with position function $\mathbf{r}(t) = t\ln t\ \mathbf{i} + t\ \mathbf{j} + e^{-t}\ \mathbf{k}$. Find the velocity, speed, and acceleration of the particle.
6. An athlete throws a shot at an angle of 45° to the horizontal at an initial speed of 43 ft/s. It leaves his hand 7 ft above the ground.
   (a) Where is the shot 2 seconds later?
   (b) How high does the shot go?
   (c) Where does the shot land?
7. Find a parametric representation for the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies between the planes $z = 1$ and $z = -1$.

Chapter 11

1. Find all second partial derivatives of $z = y \tan 2x$.
2. Find equations of the tangent plane a to the surface $z = 3x^2 - y^2 + 2x$ at the point (1,-2,1).
3. Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point (2,3,4) and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.
4. The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of the area of the triangle.
5. If $yz^4 + x^2z^3 = e^{xyz}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
6. Find the gradient of the function $f(x, y, z) = z^2e^{x\sqrt{y}}$
7. Find the directional derivative of $f(x, y, z) = x^2y + x\sqrt{1 + z}$ at the point (1,2,3) in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
8. Find the local maximum and minimum values and saddle points of the function $f(x, y) = 3xy - x^2y - xy^2$.

Chapter 12

1. Calculate $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\, dx\, dy$ by first reversing the order of integration.

2. Calculate $\iint_D (x^2+y^2)^{3/2}\, dA$, where D is the region in the first quadrant bounded by the lines $y=0$ and $y=\sqrt{3}x$ and the circle $x^2+y^2=9$

3. Calculate $\iiint_E y^2z^2\, dV$ where E is bounded by the paraboloid $x=1-y^2-z^2$ and the plane $x=0$.

4. Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=3$.

5. Consider a lamina that occupies the region D bounded by the parabola $x=1-y^2$ and the coordinate axes in the first quadrant with density function $\rho(x,y)=y$.

   (a) Find the mass of the lamina.

   (b) Find the center of mass.

   (c) Find the moments of inertia about the x- and y- axes.

Chapter 13

1. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = xy\ \mathbf{i} + x^2\ \mathbf{j}$ and C is given by $\mathbf{r}(t) = \sin t\ \mathbf{i} + (1+t)\ \mathbf{j}$, $0 \le t \le \pi$.

2. Show that $\mathbf{F}(x,y) = (1+xy)e^{xy}\ \mathbf{i} + (e^y + x^2e^{xy})\ \mathbf{j}$ is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

3. Show that $\mathbf{F}(x,y,z) = e^y\ \mathbf{i} + (xe^y + e^z)\ \mathbf{j} + ye^z\ \mathbf{k}$ is conservative and use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the line segment from (0,2,0) to (4,0,3).

4. Evaluate the surface integral $\iint_S (x^2z+y^2z)dS$ where S is the part of the plane $z = 4+x+y$ that lies inside the cylinder $x^2+y^2=4$.

5. Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x,y,z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$ where S is the part of the paraboloid $z=1-x^2-y^2$ that lies above the xy-plane and S has upward orientation.

6. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = x^3\ \mathbf{i} + y^3\ \mathbf{j} + z^3\ \mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=2$.

Solutions

Chapter 9

1. (a) 18

   (b) $\cos^{-1}\left(\frac{-1}{2\sqrt{21}}\right) \approx 96^\circ$

2. $\pm\frac{1}{3\sqrt{6}}(7\mathbf{i} + t\mathbf{j} - \mathbf{k})$

3. 6

4. $\langle 4, -3, 4\rangle$

5. $x = 1 + 3t,\ y = 2t,\ z = -1 + t$

6. $-4x + 3y + z = -14$

7. $6x + 9y - z = 26$

8. $x > y^2$

9. cylindrical: $r^2 + z^2 = 4$, spherical: $\rho^2 = 4$ or $\rho = 2$

Chapter 10

1. (a) $(-1, 0) \cup (0, 2]$

   (b) $\left\langle \sqrt{2}, 1, 0 \right\rangle$

   (c) $\left\langle \frac{1}{2\sqrt{2-t}}, \frac{te^t = e^t + 1}{t^2}, \frac{1}{t+1} \right\rangle$

2. $\frac{1}{3}\mathbf{i} - \frac{2}{\pi^2}\mathbf{j} + \frac{2}{\pi}\mathbf{k}$

3. (a) $\left(\frac{15}{8}, 0, -\ln 2\right)$

   (b) $x = 1 - 3t,\ y = 1 + 2t,\ z = t$

   (c) $3x - 2y - 2z$

4. (a) $\mathbf{T}(t) = \frac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$

   (b) $\mathbf{N}(t) = \frac{\langle 2t, 1 - t^4, -2t^3 - t\rangle}{\sqrt{t^5 + 4t^6 + 2t^4 + 5t^2}}$

   (c) $\kappa(t) = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^2}$

5. $\mathbf{v}(t) = (1+\ln t)\mathbf{i} + \mathbf{j} - e^{-t}\mathbf{k}$; $|\mathbf{v}(t)| = \sqrt{2+2\ln t + (\ln t)^2 + e^{-2t}}$; $\mathbf{a}(t) = \frac{1}{t}\mathbf{i} + e^{-t}\mathbf{k}$

6. (a) 3.8ft above the ground and 60.8 ft horizontally from athlete.
   (b) 21.4 ft
   (c) 64.2 ft from athlete

7. $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$, $0 \le \theta \le 2\pi$, $\dfrac{\pi}{3} \le \phi \le \dfrac{2\pi}{3}$

Chapter 11

1. $\dfrac{\partial z}{\partial x} = 2y\sec^2 2x$ and $\dfrac{\partial z}{\partial y} = \tan 2x$

2. tangent plane: $z = 8x + 4y + 1$ normal line: $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$

3. 38.656

4. $170cm^2$

5. $\dfrac{\partial z}{\partial x} = \dfrac{2xz^3 - yze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2z^2}$ and $\dfrac{\partial z}{\partial yx} = \dfrac{z^4 - xze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2z^2}$

6. $\nabla f = ze^{x\sqrt{y}}\left\langle z\sqrt{y}, \dfrac{xz}{2\sqrt{y}}, 2\right\rangle$

7. $D_u f(1,2,3) = \dfrac{25}{6}$

8. (0,0), (3,0), (0,3) are saddle points and (1,1) is a local maximum

Chapter 12

1. $\dfrac{1}{4}(e-1)$

2. $\dfrac{81\pi}{5}$

3. $\dfrac{\pi}{96}$

4. 12π

5. (a) 1/4
   (b) $(\bar{x}, \bar{y}) = \left(\dfrac{1}{3}, \dfrac{8}{15}\right)$
   (c) $I_x = 1/12$, and $I_y = 1/24$

Chapter 13

1. $\dfrac{\pi}{4}$
2. $f(x, y) = e^y + xe^{xy} + K$
3. 2
4. $32\pi\sqrt{3}$
5. Since curl $\mathbf{F} = \mathbf{0}$, $\displaystyle\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = 0$. We parameterize C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le 2\pi$ and $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t)\, dt = 0$
6. 11π