

Directions: Answer the following questions. You must SHOW ALL WORK. You are allowed to use a nonprogrammable, nongraphing calculator. No cell phones, computers or PDAs. There are a total of **100 points** available.

1. Suppose a colony of ants develops according to the logistic equation. Suppose the colony begins at 10,000 ants and the maximum number of ants that can be supported is 100,000 ants. Further suppose that after 2 days there are 50,000 ants in the colony.

(a) State both the differential equation that models this situation as well as its solution. [6 pts]

Solution

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right); \quad P(t) = \frac{K}{Ae^{-kt} + 1} \text{ where } A = \frac{K - P_0}{P_0}$$

(b) How many ants are present when 6 days have past? [6 pts]

Solution

From the word problem we know the following: $P(0) = P_0 = 10,000$; $K = 100,000$; $P(2) = 50,000$. Before forming the analytic solution we must first solve for k .

$$\begin{aligned} 50000 &= \frac{100000}{9e^{-2k} + 1} \\ 9e^{-2k} + 1 &= 2 \\ 9e^{-2k} &= 1 \\ e^{-2k} &= 1/9 \\ k &= \frac{-1}{2} \ln \frac{1}{9} = \ln(3) \approx 1.0986 \end{aligned}$$

So at time t , the population is given by $P(t) = \frac{100000}{9e^{1/2 \ln(1/9)t} + 1}$.

So $P(6) = \frac{100000}{9e^{3 \ln(1/9)} + 1} \approx 98780$ ants.

2. Solve the initial value problem $y'' - 2y' - 3y = 0$, $y(0) = -1$ and $y'(0) = 5$. [12 pts]

Solution

The characteristic equation is given by $r^2 - 2r - 3 = 0$ which gives us roots of $r_1 = 3$ and $r_2 = -1$. Since there are two real, unequal roots the complementary solution is given by

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$

Using the ICs we can solve for c_1 and c_2 .

$$\begin{aligned} y(x) &= c_1 e^{3x} + c_2 e^{-x}; y(0) = -1 \Rightarrow c_1 + c_2 = -1 \\ y'(x) &= 3c_1 e^{3x} - c_2 e^{-x}; y'(0) = 5 \Rightarrow 3c_1 - c_2 = 5 \end{aligned}$$

which allows us to conclude that $c_1 = 1$ and $c_2 = -2$. Therefore the solution to the differential equation is given by

$$y(x) = e^{3x} - 2e^{-x}$$

3. Using undetermined coefficients set up your guess for the particular solution to the following differential equation. DO NOT SOLVE. [8 pts]

$$y'' + y' = e^{-x} \sin(3x)$$

Solution

The characteristic equation is given by $r^2 + r = 0$ which gives us roots of $r_1 = 0$ and $r_2 = -1$. Since the roots are real and complex the complementary solution is given by

$$y_c(x) = c_1 + c_2 e^{-x}$$

The particular solution will look like

$$y_p(x) = A e^{-x} \sin(3x) + B e^{-x} \cos(3x)$$

4. Solve $y'' + 2y' + y = x^2$ by using the method of undetermined coefficients. Clearly show each step. [12 pts]

Solution

The characteristic equation is given by $r^2 + 2r + 1 = 0$ which gives us a root of $r = -1$. Since the root is a repeated root then the complementary solution is given by

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$$

The particular solution will look like

$$\begin{aligned} y_p(x) &= Ax^2 + Bx + C \\ y_p'(x) &= 2Ax + B \\ y_p''(x) &= 2A \end{aligned}$$

Plugging these derivatives into the original equation will give

$$2A + 4Ax + 2B + Ax^2 + Bs + C = x^2$$

By equating coefficients we get $A = 1$, $B = -4$, $C = 6$. Therefore the particular solution is given by

$$y_p(x) = x^2 - 4x + 6$$

So the solution to the differential equation is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + x^2 - 4x + 6$$

5. A spring with a mass of m kg has damping constant 30 and spring constant 45.

- (a) Find the mass that would produce critical damping. [5 pts]

Solution

To produce critical damping $c^2 - 4mk = 0$.

$$\text{So } 30^2 - 4m45 \Rightarrow m = 5$$

- (b) Use the value you find for m in the first part to state the position function $x(t)$ of the mass after t seconds. Make sure your independent variable and dependent variable are correct. [11 pts]

Solution

The differential equation is given by

$$5x'' + 30x' + 45x = 0$$

The characteristic equation is $5r^2 + 30r + 45 = 0$ which gives us a repeated root of $r = -3$. Therefore the solution is

$$x(t) = c_1e^{-3t} + c_2te^{-3t}$$

6. Write an expression for a_n of the sequence [10 pts]

$$\left\{ \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots \right\}$$

Solution

$$a_n = \frac{n+1}{n+2}$$

7. Determine whether the sequence is convergent or divergent. If it is convergent find its limit. [10 pts each]

(a) $a_n = \frac{3^{n+1}}{4^n}$

Solution

$$a_n = \frac{3^{n+1}}{4^n} = \frac{3^n 3}{4^n} = 3 \left(\frac{3}{4} \right)^n$$

Since $r = \frac{3}{4} < 1$ Then by the theorem this converges to 0.

(b) $a_n = \frac{(n-2)!}{n!}$

Solution

$$a_n = \frac{(n-2)!}{n!} = \frac{(n-2)!}{n(n-1)(n-2)!} = \frac{1}{n(n-1)}$$

So as $n \rightarrow \infty$ the sequence converges to 0.

(c) $a_n = \frac{\cos \pi n}{n^2}$

Solution

$$\frac{-1}{n^2} \leq \frac{\cos \pi n}{n^2} \leq \frac{1}{n^2}$$

We know that $\lim_{n \rightarrow \infty} \frac{-1}{n^2} = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$. Therefore by the Squeeze Theorem we can say that a_n converges to 0.