Test #4 Answer Key

1a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ Alternating Series Test w/ $a_n = \frac{1}{\sqrt{n+1}}$

$a_{n+1} < a_n \Rightarrow \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}}$ ✓

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ✓ ∴ so

∴ $\boxed{\text{series converges}}$

b) $\sum_{n=1}^{\infty} \frac{n^2}{3n^2+1}$ Test for Divergence $\lim_{n\to\infty} \frac{n^2}{3n^2+1} = \frac{1}{3} \neq 0$

∴ $\boxed{\text{series diverges!}}$

c) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ Ratio Test

$\lim_{n\to\infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{e} \right| = \infty \Rightarrow \boxed{\text{Diverges}}$

2. $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$ ① Rewrite as partial fractions

$\frac{A}{(2n+1)} + \frac{B}{(2n-1)} = \frac{2}{4n^2-1} \Rightarrow A(2n-1) + B(2n+1) = 2$

$2An - A + 2Bn + B = 2$

$2A + 2B = 0 \qquad -A+B = 2$

$A + B = 0 \qquad A+B = 0$

$2B = 2$

$B = 1 \quad A = -1$

$= \sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1}$

② Calculate n^{th} partial sum

$S_n = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{1}{2n-1} - \frac{1}{2n+1}$

$S_n = 1 - \frac{1}{2n+1}$

$\lim_{n\to\infty} 1 - \frac{1}{2n+1} = \boxed{1}$

③ $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ $\frac{n}{n^4+1} < \frac{n}{n^4} = \frac{1}{n^3}$, $\frac{1}{n^3}$ converges by p-series test

∴ $\boxed{\text{series converges}}$ ← Direct Comparison Test

OR $\lim_{n\to\infty} \frac{\frac{n}{n^4+1}}{\frac{1}{n^3}} \overset{\lim_{n\to\infty}}{=} \frac{n^4}{n^4+1} = 1 \Rightarrow \boxed{\text{converges}}$ since $\frac{1}{n^3}$ converges

Limit Comparison Test

④ $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = \frac{8}{5} + \frac{32}{25} + \frac{128}{125} + \dots$

$\Rightarrow a = \frac{8}{5} \quad r = \frac{4}{5}$ since $|r| < 1 \Rightarrow$ geometric series | conver

to $s = \frac{a}{1-r} = \boxed{8}$

5) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$ ① Ratio Test

$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{(x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+2)n}{4(n+1)} \right| = \frac{|x+2|}{4} < 1$

$\Rightarrow$ convergence when $|x+2| < 4 \Rightarrow \boxed{R = 4}$

$|x+2| < 4 \Rightarrow -4 < x+2 < 4 \Rightarrow -6 < x < 2$

② Check convergence @ endpoints

@ $x = -6$ $\quad \sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n}$ converges by Alternating Series Test

@ $x = 2$ $\quad \sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by harmonic series

$\therefore \boxed{I = -6 \leq x < 2 = [-6, 2)}$

6) $f(x) = \int \frac{1}{x-5}\,dx$

$= \int \frac{1}{-5+x}\,dx = \int \frac{1}{-5(1-\frac{x}{5})}\,dx = \int -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = -\frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^n}$

$= \boxed{-\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}}$