

MA-241
Test 4B Answer Key

1. (a) Use the Test for Divergence:

$$\lim_{n^2 \rightarrow \infty} \frac{n}{2n^2 + 3} = \frac{1}{2} \neq 0$$

Therefore, the series, $\sum_{n=1}^{\infty} \frac{n^2}{3n^2 + 1}$ is divergent.

- (b) Use the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{2} \right| = \infty$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ is divergent.

2. First rewrite $\frac{2}{(2n+1)(2n+3)}$ as a partial fraction.

$$= \frac{A}{2n+1} + \frac{B}{2n+3} \implies A = 1, B = -1$$

Therefore, $\frac{2}{(2n+1)(2n+3)} = \frac{1}{2n+1} - \frac{1}{2n+3}$.

Next calculate the n^{th} partial sum

$$s_n = \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{9} \right) + \cdots + \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right)$$
$$s_n = \frac{1}{3} - \frac{1}{2n+3}$$

$$\lim_{n \rightarrow \infty} 1 - \frac{1}{2n+3} = \frac{1}{3}$$

3. $\frac{2n^2}{3n^5 + 2n + 1} < \frac{2n^2}{3n^5} = \frac{2}{3n^3}, \frac{1}{n^3}$ converges by p -series test. Therefore the series $\sum_{n=1}^{\infty} \frac{2n^2}{3n^5 + 2n + 1}$ converges by the Comparison Test.

4. $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = \frac{8}{5} + \frac{32}{25} + \frac{128}{125} + \cdots$
 $\implies a = \frac{8}{5}, r = \frac{4}{5}$

Since $|r| < 1 \implies$ geometric series converges to $\frac{a}{1-r} = 8$

5. First apply the Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{(x+2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+2)n}{4(n+1)} \right| = \frac{|x+2|}{4} < 1$$

\Rightarrow convergence when $|x+2| < 4$. So $R = 4$.

So far we know that $I = -6 < x < 2$.

The next step is to check the endpoints

At $x = -6$

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n4^n} = \frac{(-1)^n}{n}$$

which converges by the Alternating Series Test.

At $x = 2$

$$\sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges by harmonic series.

Therefore $I = -6 \leq x < 2$.

$$\begin{aligned} 6. f(x) &= \int \frac{1}{x^2-4} dx = \int \frac{1}{-4+x^2} dx = \int \frac{1}{-4\left(1-\frac{x^2}{4}\right)} dx = -\frac{1}{4} \int \frac{1}{1-\frac{x^2}{4}} dx \\ &= \int -\frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{x^2}{4}\right)^n dx = \int -\frac{1}{4} \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n} dx = -\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)4^{n+1}} \end{aligned}$$

7. (a) The Maclaurin series for e^x

$$\begin{aligned} f(x) &= e^x, f(0) = 1 \\ f'(x) &= e^x, f'(0) = 1 \\ f''(x) &= e^x, f''(0) = 1 \\ &\vdots, \vdots \end{aligned}$$

Therefore the Maclaurin series for $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$(b) f(x) = xe^{-3x} = x \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{n+1}}{n!}$$