MA-241
Test 4 Answer Key

1. (a) Use the Test for Divergence:

$$\lim_{n\to\infty} \frac{n^2}{3n^2+1} = \frac{1}{3} \neq 0$$

Therefore, the series, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3n^2+1}$ is divergent.

(b) Use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{(n+1)}{e}\right| = \infty$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$ is divergent.

2. First rewrite $\dfrac{2}{(2n+1)(2n-1)}$ as a partial fraction.

$$= \frac{A}{2n+1} + \frac{B}{2n-1} \Longrightarrow A = -1, B = 1$$

Therefore, $\dfrac{2}{(2n+1)(2n-1)} = \dfrac{1}{2n-1} - \dfrac{1}{2n+1}$.
Next calculate the n^{th} partial sum

$$s_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$
$$s_n = 1 - \frac{1}{2n+1}$$

$$\lim_{n\to\infty} 1 - \frac{1}{2n+1} = 1$$

3. $\dfrac{n}{n^4+1} < \dfrac{n}{n^4} = \dfrac{1}{n^3}, \dfrac{1}{n^3}$ converges by p-series test. Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges by the Comparison Test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = \frac{8}{5} + \frac{32}{25} + \frac{128}{125} + \cdots$
$\Rightarrow a = \dfrac{8}{5}, r = \dfrac{4}{5}$
Since $|r| < 1 \Rightarrow$ geometric series converges to $\dfrac{a}{1-r} = 8$

5. First apply the Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{(n+1)6^{n+1}}\cdot\frac{n6^n}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)n}{6(n+1)}\right| = \frac{|x+3|}{6} < 1$$

$\Rightarrow$ convergence when $|x+3| < 6$. So $R = 6$.
So far we know that $I = -9 < x < 3$.
The next step is to check the endpoints
At $x = -9$

$$\sum_{n=1}^{\infty}\frac{(-6)^n}{n6^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-6)^n}{n6^n} = \frac{(-1)^n}{n}$$

which converges by the Alternating Series Test.
At $x = 3$

$$\sum_{n=1}^{\infty}\frac{6^n}{n6^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges by harmonic series.
Therefore $I = -9 \le x < 3$.

6. $f(x) = \int \frac{1}{x^2-5}dx = \int \frac{1}{-5+x^2}dx = \int \frac{1}{-5\left(1-\frac{x^2}{5}\right)}dx = -\frac{1}{5}\int \frac{1}{1-\frac{x^2}{5}}dx$

$$= \int -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x^2}{5}\right)^n dx = -\sum_{n=0}^{\infty}\frac{x^{2n}}{5^n}dx = -\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(n+1)5^{2n+1}}$$

7. (a) The Maclaurin series for e^x

$f(x) = e^x$, $f(0) = 1$
$f'(x) = e^x$, $f\prime(0) = 1$
$f''(x) = e^x$, $f\prime\prime(0) = 1$
$\vdots$, $\vdots$

Therefore the Maclaurin series for $f(x) = e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

(b) $f(x) = xe^{2x} = x\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n x^{n+1}}{n!}$