Practice Test 1

MA 522 Fall 2011

1. ***Correctness of Horner's rule***
The following short pseudo-code implements Horner's rule for evaluating a polynomial

$$
\begin{aligned}
P(x) &= \sum_{i=0}^{n} a_i x^i \\
&= a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + x a_n)\cdots)) \ \in R[x]
\end{aligned}
$$

where R is a ring.

Input $a_0, a_1, \ldots, a_n \in R$ and a value $x^* \in R$.

Output $P(x^*) = \sum_{i=0}^{n} a_i (x^*)^i$

```
1   y ← 0
2   i ← n
3   while i ≥ 0 do
4       y ← a_i + x* · y
5       i ← i − 1
6   return(y)
```

(a) What is the asymptotic running time (arithmetic operations over R) of this code for Horner's rule?

(b) The naive polynomial-evaluation algorithm computes each term of the polynomial from scratch. What is the asymptotic running time of this algorithm?

(c) Prove that the following is a loop invariant for the **while** loop in lines 3-5, and conclude that the given code correctly evaluates a polynomial:

> At the start of each iteration of the **while** loop of lines 3-5,
> $$y = \sum_{k=0}^{n-(i+1)} a_{k+i+1} (x^*)^k.$$

2. The ***Fibonacci numbers*** are defined by the following recurrence:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_i &= F_{i-1} + F_{i-2} \quad \text{for } i \geq 2 \end{aligned}$$

(a) Show (by induction on k) that for $k \geq 1$ we have $\gcd(F_{k+1}, F_k) = 1$ and that the Euclidean Algorithm takes exactly k iterations on input F_{k+1}, F_k.

(b) Prove by induction that s, t satisfies

$$sF_{k+1} + tF_k = 1$$

when $s = \pm F_{k-1}$ and $t = \mp F_k$.

3.

(a) Compute $(3^{-1} \mod 14)$ and $(14^{-1} \mod 3)$.

(b) Assume that $x > 0$ satisfies the quadratic equation $x^2 - 5x - 1 = 0$. Give the infinite continued fraction representation of x, preferably without the use of calculators.

4. Carl, Joachim and Jurgen met at a party on Thursday, December 31st, 1998. They were trying to get together in the New Year, but they were very busy people. Carl was busy, except on Fridays, Joachim had time on January 7th and then again on every 9th day, and Jurgen was free on January 6th and then again every 11th day. Which was the earliest date that they could meet?

5. Let $\dfrac{r(x)}{t(x)}$ be the (n, m)-Padé approximant to the function $f(x)$. Show that the $(n+k, m)$-Padé approximant to $x^k f(x)$ is $\dfrac{x^k r(x)}{t(x)}$.