HOMEWORK 2 SOLUTIONS
MA 520

2.17. (a) (3 points) The elements of \( \mathcal{F}(\mathbb{R}^2, \mathbb{R}^2) \) are (ii) \( \begin{bmatrix} x - y \\ x \cdot y \end{bmatrix} \), (iii) \( e^x \cos(y) \), and (iv) \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \).

(b) (2 points)
\[
\begin{bmatrix}
  x - y + e^x + 1 \\
  x \cdot y + \cos(y) + 3
\end{bmatrix} =
\begin{bmatrix}
  -5 \cdot x - 5 \cdot y - 5 \cdot e^x - 5 \\
  -5 \cdot x \cdot y - 5 \cdot \cos(y) - 15
\end{bmatrix}
\]

(c) (2 points) The zero element is the zero function, which for every vector \( \begin{bmatrix} x \\ y \end{bmatrix} \) in \( \mathbb{R}^2 \) has function value \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) in \( \mathbb{R}^2 \).

2.1.14. (5 points) Let \( V := \{ (f(x), a) \mid f: \mathbb{R} \to \mathbb{R} \text{ continuous, } a \in \mathbb{R} \} \). The addition in \( V \) is \( (f(x), a) + (g(x), b) = (f + g)(x), a + b) \). The scalar multiples \( c \cdot (f(x), a) = (cf(x), c \cdot a) \).

Then \( V = C^0(\mathbb{R}) \times \mathbb{R} \) and since both \( C^0(\mathbb{R}) \) and \( \mathbb{R} \) and vector spaces over \( \mathbb{R} \), their Cartesian product is also a vector space by exercise 2.1.13.

The zero element is \( (0(x), 0) \), where the first zero is the constant function, taking zero value for all \( x \in \mathbb{R} \).

2.2.6. (a) (4 points) For example, the union of the two lines \( \{ (x, 0) \mid x \in \mathbb{R} \} \cup \{ (0, y) \mid y \in \mathbb{R} \} \subseteq \mathbb{R}^2 \) is closed under scalar multiplication but not under vector addition.

(b) (4 points) For example the nonnegative quadrant \( Q = \{ (x, y) \mid x, y \geq 0 \} \subseteq \mathbb{R}^2 \) is closed under addition, but not under scalar multiplication.

2.2.14. (5 points)
Let \( S = \{ f \in C^0(\mathbb{R}) \mid f(1) = 0 \} \). Then if \( f, g \in S \), i.e. \( F(1) = g(1) = 0 \), then \( (f + g)(1) = 0 \) and also \( (cf)(1) = 0 \), so \( S \) is a subspace of \( C^0(\mathbb{R}) \).

Let \( T = \{ f \in C^0(\mathbb{R}) \mid f(0) = 1 \} \). For example, 1 and \( e^x \) are both in \( T \), but \( (e^x + 1)(0) = 2 \), so their sum is not in \( T \), so \( T \) is not a subspace.

In general \( \{ f \in C^0(\mathbb{R}) \mid f(a) = b \} \) is a subspace of \( C^0(\mathbb{R}) \) if and only if \( b = 0 \).

2.3.7. (a) (4 points) Let \( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \) be a general symmetric matrix. Then we have \( \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \), so the 3 matrices span the space of symmetric matrices.
2.3.9. (6 points) (a) yes, (b) no, (c) no, (d) yes: \( \cos^2(x) = 1 - \sin^2(x) \), (e) no, (f) no.

2.3.22. (a) (5 points) To show that the 3 vectors are linearly independent, we need to show that the homogeneous linear system

\[
\begin{bmatrix}
1 & -2 & 2 \\
0 & 3 & -2 \\
2 & -1 & 1 \\
1 & -1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

has a unique solution.

Since
\[
\begin{bmatrix}
1 & -2 & 2 \\
0 & 3 & -2 \\
2 & -1 & 1 \\
1 & -1 & -1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 2 \\
0 & 3 & -2 \\
0 & 0 & -1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

so its rank is equal to the number of columns, thus the solution is unique, and the 3 vectors are linearly independent.

(b) (5 points) We will find out if any of the 4 vectors are in the span by computing the row echelon form of the matrix (obtained by adding the 4 column vectors to the end of the matrix in part (a)).

\[
\begin{bmatrix}
1 & -2 & 2 & 1 & 1 & 0 & 0 \\
0 & 3 & -2 & 1 & 0 & 1 & 0 \\
2 & -1 & 1 & 2 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 2 & 1 & 1 & 0 & 0 \\
0 & 3 & -2 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & -1 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

This shows that the vectors in (i), (iii) and (iv) are in the span.

(c) (5 points) We compute the row echelon form of the following matrix:

\[
\begin{bmatrix}
1 & -2 & 2 & a \\
0 & 3 & -2 & b \\
2 & -1 & 1 & c \\
1 & -1 & 1 & d \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 2 & a \\
0 & 3 & -2 & b \\
0 & 0 & -1 & c - 2a - b \\
0 & 0 & 0 & d + a - c \\
\end{bmatrix}
\]

which shows that \( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \) is in the span iff \( d + a - c = 0 \).
2.3.31. \( (a) \) (4 points) Suppose there are \( c_1, \ldots, c_k \) in \( \mathbb{R} \) such that \( \sum_{i=1}^{k} c_i v_i = 0 \). Then defining \( c_k + 1 \) = 0, \ldots, \( c_m = 0 \) we have that \( \sum_{i=1}^{m} c_i v_i = 0 \), but since \( v_1, \ldots, v_m \) are linearly independent, we must have \( c_1 = \ldots = c_k = 0 \).

\( (b) \) (2 points) No, for example, \( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \), \( \left[ \begin{array}{c} 2 \\ 2 \end{array} \right] \) are dependent, but \( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \) alone is independent.

2.3.37. \( (a) \) (5 points) \( \textbf{Claim} \): If the sample vectors \( f_1 = \ldots, f_k \in \mathbb{R}^n \) are linearly independent then the functions \( f_1, \ldots, f_k \in C^0(\mathbb{R}) \) are also independent.

\( \textbf{Proof:} \) Assume that \( f_1, \ldots, f_k \in \mathbb{R}^n \) are linearly independent Let \( c_1, \ldots, c_k \in \mathbb{R} \) such that \( c_1 f_1 + \ldots + c_k f_k = 0 \) as functions. That means that for all \( x \in \mathbb{R} \) the evaluation \( c_1 f_1(x) + \ldots + c_k f_k(x) = 0 \). In particular it is true for the sample points \( x_1, \ldots, x_n \), so we have \( c_1 f_1(x_i) + \ldots + c_k f_k(x_i) = 0 \) for all \( i = 1 \ldots n \). Writing this in a vector form we get that

\[
\begin{bmatrix}
  f_1(x_1) \\
  \vdots \\
  f_1(x_n)
\end{bmatrix}
+ 
\begin{bmatrix}
  f_k(x_1) \\
  \vdots \\
  f_k(x_n)
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  \vdots \\
  0
\end{bmatrix},
\]

\( i.e. \ c_1 f_1 + \ldots + c_k f_k = 0 \). But since \( f_1, \ldots, f_k \) are linearly independent, we must have \( c_1 = \ldots = c_k = 0 \). This implies that \( f_1, \ldots, f_k \) are linearly independent.

\( (b) \) (2 points) Let \( n = k = 2 \), \( x_1 = 1, x_2 = -1 \), \( f_1 = x, f_2 = x^2 \), then \( f_1 \) and \( f_2 \) are independent over \( \mathbb{R} \), but both of the evaluation vectors are equal to \( \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \), so these are dependent.

\( (c) \) (4 points) Let \( x_1 = 0, x_2 = \frac{\pi}{4}, x_3 = \frac{\pi}{2}, x_4 = \pi, x_5 = -\frac{3}{4} \pi \). Then the evaluation vectors form the following matrix:

\[
\begin{bmatrix}
  1 & \cos(0) & \sin(0) & \cos(0) & \sin(0) \\
  1 & \cos\left(\frac{1}{4} \pi\right) & \sin\left(\frac{1}{4} \pi\right) & \cos\left(\frac{1}{2} \pi\right) & \sin\left(\frac{1}{2} \pi\right) \\
  1 & \cos\left(\frac{1}{2} \pi\right) & \sin\left(\frac{1}{2} \pi\right) & \cos(\pi) & \sin(\pi) \\
  1 & \cos(\pi) & \sin(\pi) & \cos(2 \pi) & \sin(2 \pi) \\
  1 & \cos\left(-\frac{3}{4} \pi\right) & \sin\left(-\frac{3}{4} \pi\right) & \cos\left(-\frac{3}{2} \pi\right) & \sin\left(-\frac{3}{2} \pi\right)
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
  1 & 1 & 0 & 1 & 0 \\
  1 & \frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{2} & 0 & 1 \\
  1 & 0 & 1 & -1 & 0 \\
  1 & -1 & 0 & 1 & 0 \\
  1 & -\frac{1}{2} \sqrt{2} & -\frac{1}{2} \sqrt{2} & 0 & 1
\end{bmatrix}
\]

The determinant is \( 4 \sqrt{2} \), so the 5 evaluation vectors are linearly independent, which implies that \( \{ 1, \ldots, 5 \} \)
\(\cos(x)\), \(\sin(x)\), \(\cos(2x)\), \(\sin(2x)\) are linearly independent

2.4.3. There are \(\infty\) many choices for bases, you have to check that they satisfy the equations, independent, and their number is the dimension.

\[(a)\ (3\ points)\ \left\{ \begin{array}{l}
x \\
y \\
z \\
\end{array} \right\} \in \mathbb{R}^3: z - 2y = 0 = \text{span} \left\{ \begin{array}{l}
1 \\
0 \\
0 \\
\end{array} \right\},
\]

\[(b)\ (3\ points)\ \left\{ \begin{array}{l}
x \\
y \\
z \\
\end{array} \right\} \in \mathbb{R}^3: 4x + 3y - z = 0 = \text{span} \left\{ \begin{array}{l}
-\frac{3}{4} \\
1 \\
0 \\
\end{array} \right\},
\]

\[(c)\ (3\ points)\ \left\{ \begin{array}{l}
x \\
y \\
z \\
w \\
\end{array} \right\} \in \mathbb{R}^4: x + 2y + z - w = 0 = \text{span} \left\{ \begin{array}{l}
-2 \\
1 \\
0 \\
0 \\
\end{array} \right\},
\]

2.4.8. (a) (5 points)

\[
\left\{ \begin{array}{l}
x \\
y \\
z \\
w \\
\end{array} \right\} \in \mathbb{R}^4: \begin{bmatrix}
1 & 2 & -1 & 1 \\
3 & 0 & 2 & -1 \\
\end{bmatrix} \cdot \left\{ \begin{array}{l}
x \\
y \\
z \\
w \\
\end{array} \right\} = \text{span} \begin{bmatrix}
1 & 2 & -1 & 1 \\
0 & -6 & 5 & -4 \\
\end{bmatrix},
\]

\[
\text{Dimension is 2.}
\]

(b) (5 points) \(p(x) = ax^2 + bx + c : a, b, c \in \mathbb{R}, p(1) = 0\) = \(p(x) = ax^2 + bx + c : a, b, c \in \mathbb{R}, a + b + c = 0\) = \(\text{span}\{x^2 - x, x^2 - 1\}\), Dimension is 2.

(c) (5 points) First solve \(\lambda^3 - \lambda^2 + 4\lambda - 4 = (\lambda - 1) (\lambda^2 + 4) = 0\), the solution are \(\lambda = 1, \lambda = 2 i, \lambda = -2 i\).

\(\{u \in C^\infty(\mathbb{R}) : u'' - u + 4u' - 4u = 0\} = \text{span}(e^x, \sin(2x), \cos(2x))\), Dimension is 3.
2.4.15. (5 points) Since the dimension of $M_{2 \times 2}$ is 4, we need to find the values of $k$ when the 4 given matrix is linearly independent. We can "flatten" the matrices into vectors in $\mathbb{R}^4$:

$$A_1 \mapsto \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, A_2 \mapsto \begin{bmatrix} k \\ -3 \\ 1 \\ 0 \end{bmatrix}, A_3 \mapsto \begin{bmatrix} 1 \\ 0 \\ -k \\ 2 \end{bmatrix}, A_4 \mapsto \begin{bmatrix} 0 \\ k \\ -1 \\ -2 \end{bmatrix}.$$ 

Then we put these column vectors into a matrix, and compute the determinant:

$$\det \begin{bmatrix} 1 & k & 1 & 0 \\ -1 & -3 & 0 & k \\ 0 & 1 & -k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} = -2k - 4 + 2k^2$$

This is not zero if and only if $k \neq 2$ and $k \neq -1$. 

>