Computing tropical curves via homotopy continuation

arXiv:1408.3105

Josephine Yu

joint work with
Anders Jensen and Anton Leykin

FoCM ’14
Computational Algebraic Geometry
December 20, 2014
Suppose we have a set of lattice points $A \subset \mathbb{Z}^n$ and want to compute the **volume of its convex hull**.

Suppose our computer does not have any convex hull or Gröbner basis algorithms installed, but we do have **numerical solvers**.

What do we do?
Suppose we have a set of lattice points \( A \subset \mathbb{Z}^n \) and want to compute the **volume of its convex hull**.

Suppose our computer does not have any convex hull or Gröbner basis algorithms installed, but we do have **numerical solvers**.

What do we do?

**Theorem (Kushnirenko 1975)**

Let \( f_1, \ldots, f_n \) be polynomials with generic coefficients, each with support \( A \). Then the number of common zeroes in \((\mathbb{C}^*)^n\) is equal to \( n! \) \( \text{vol}(\text{conv}(A)) \).

Similarly we can use Bernstein’s Theorem to compute mixed volumes.
Now, compute the facet normals of the convex hull.
(Again, no convex hull algorithms, no Gröbner bases.)
Now, compute the facet normals of the convex hull. (Again, no convex hull algorithms, no Gröbner bases.)

Our numerical algorithm gives us all the facet normals and facet volumes without using any of the usual convex hull techniques.

Let $f_1, \ldots, f_{n-1}$ be $n - 1$ polynomials in $n$ variables with generic coefficients, each with support $A$. Then the tropical curve of the ideal $\langle f_1, \ldots, f_{n-1} \rangle$ consists of rays in facet normal directions of $\text{conv}(A)$, with multiplicities equal to the normalized facet volumes.
Tropical Curves

Computational Problem:

Input: ideal \( I \) defining a curve in \( (\mathbb{C}^*)^n \)

Output: tropical curve \( \mathcal{T}(I) \)

Let \( I \) be in ideal in \( \mathbb{C}[x] \) defining a curve \( \mathbb{V}(I) \) in \( (\mathbb{C}^*)^n \).

The **tropical curve** \( \mathcal{T}(I) \) of \( I \) is

- limit of amoeba: \( \lim_{t \to \infty} \{ \log_t |x| : x \in \mathbb{V}_\mathbb{C}(I) \} \)
- valuation of \( \mathbb{K} \) (Puiseux-series) points: \( \{ \text{val}(x) : x \in \mathbb{V}_\mathbb{K}(I) \} \)
- monomial-free degenerations:
  \[ \{ w \in \mathbb{R}^n : \text{in}_w(I) \text{ contains no monomials} \} \]

We want to do all the computations Gröbner-free.
Assume that we can compute numerical approximations of complex solutions of polynomial equations, but we are not able to compute a single Gröbner basis of the ideal \( I \).
Related works


In theory, we can (numerically) project the curve onto coordinate 2-planes, find the Newton polygons, then patch these together to get the big tropical curve.

In practice this would involve working with very large degree polynomials, which is undesirable for numerical computations.
Our method

Input: generators of an ideal $I$ defining a curve in $(\mathbb{C}^*)^n$

Output: tropical curve $\mathcal{T}(I)$

Step 0: Compute the degree of the curve which can be done by slicing with a generic hyperplane.

Step 1: Guess some rays by slicing amoeba with hyperplanes in log-coordinates

Step 2: Compute multiplicities by computing constant terms of Puiseux series solutions to some polynomial equations (ray is in the tropical curve $\iff$ multiplicity $> 0$)

Step 3: Check if we found all rays by computing the degree of the found rays tropically

Repeat: Go back to Step 1 if necessary.
Step 0: Compute the degree of the curve

- We can compute the degree of the curve in \((\mathbb{C}^*)^n\) by slicing with a generic hyperplane and counting the number of intersection points.
- The degree is an **upperbound on the absolute value of integers** appearing in the primitive integer directions along **rays** of the tropical curve.
Step 1: Guess some rays

- By computing approximate zeroes of $I + \langle x^a - C \rangle$, we get some points in the intersection of the amoeba with a usual hyperplane defined by $X \cdot a = \log(|C|)$.
- Change the $C$ so that hyperplane moves away from the origin, and numerically track the zeroes.
- Find a small integer vector in approximate directions of paths.
Step 2: Compute multiplicities

Now we want to check if a given point belongs to a tropical variety.

First reduce to the case when the tropical variety (and the original variety) is zero dimensional, by slicing the tropical curve transversely with an affine linear space at the given point.
Step 2: Compute multiplicities

Let $\mathbb{K}$ be the field of Puiseux series in one variable $t$ with complex coefficients, convergent in a punctured neighborhood of 0 in the complex plane.

For a zero dimensional ideal $J$ over $\mathbb{K}$, the **multiplicity** of a point $w$ in $\mathcal{T}(J)$ is the number of $\mathbb{K}$-zeroes of $J$ with degree $w$.

**Problem:** Given $w \in \mathbb{R}^n$ and zero dimensional ideal $J$ in $\mathbb{K}[x]$, compute the multiplicity of $w$ in $\mathcal{T}(J)$.

($w \in \mathcal{T}(J) \iff \text{multiplicity} > 0.$)

- By substituting $x_i$ with $x_i t^{-w_i}$ we may assume that $w = 0$. We want to count the number of (convergent) Puiseux series solutions with non-zero constant term.
- We can plug in a small number for $t$, compute complex solutions, and track the solutions as $t \to 0$.
- Count the number of homotopy paths that converge to a point with finite non-zero coordinates.
Step 3: Check if we found all rays

- For each of the found rays, write it (in a minimal way) as a positive linear combination of \(-e_0, -e_1, \ldots, -e_n\) where 
  \(e_0 := -(e_1 + \cdots + e_n)\).

- The balancing condition is satisfied if and only if each \(-e_j\) is used the same number of times.

- This number is the degree of the tropical curve.

- If this is not the same as the degree we computed at the beginning, then go back and find some more rays.

**Termination:** The degree of the curve gives a bound on the integer coordinates of rays in the tropical curve. There are only finitely many such rays. So the guessing and checking cannot go on forever.
Implementation

- We implemented in Macaulay2.
- Code is posted on my website.
- We use NAG4M2, Bertini, and PHCPack.
Application to knot theory

- For every knot, there is a knot invariant called the $A$-polynomial, in two variables. It is a generalization of Alexander polynomial.
- The boundary slopes of its Newton polygons are of interest to knot theorists, as they have geometric.
- The plane curve defined by $A$-polynomial is a projection (under a monomial map) of a curve in high dimension, whose equations are easy to find.
- Our numerical methods can compute those boundary slopes in some cases where other methods could not. (9-crossing knots, ambient dimension $\sim 30$, degree $\sim 450$)
Future directions

- higher dimensional varieties
- certification
- hybrid methods
Future directions

- higher dimensional varieties
- certification
- hybrid methods

Thank you for your attention!