Cactus Varieties of Cubic Forms

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Report on

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KR
Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $n$.

**Definition**

The $r$-th secant variety $\text{Sec}_r(X)$ is the closure of the union of $r - 1$-dimensional linear spaces $L \subset \mathbb{P}^N$ that intersect $X$ in $r$ linearly independant points.

$$\dim \text{Sec}_r(X) \leq \min\{nr + r - 1, N\}.$$
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**Definition**

The $r$-th cactus variety $Cactus_r(X)$ is the closure of the union of $r - 1$-dimensional linear spaces $L$ in $\mathbb{P}^N$ that intersect $X$ in a scheme of length $r$ that spans $L$. Clearly $Sec_r(X) \subset Cactus_r(X)$.

When is $Sec_r(X) \neq Cactus_r(X)$? What is $\dim Cactus_r(X)$?
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Clearly $\text{Sec}_r(X) \subset \text{Cactus}_r(X)$.

When is $\text{Sec}_r(X) \neq \text{Cactus}_r(X)$? What is $\dim \text{Cactus}_r(X)$?
Let \( X = X_{d,n} = \{ l^d | l \in \langle x_0, \ldots, x_n \rangle \} \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d) \).

If \( d = 2 \) or \( n \leq 3 \), then

\[
\text{Sec}_r(X_{d,n}) = \text{Cactus}_r(X_{d,n})
\]
We specialize to

$$X = X_{3,n} = \{l^3 | l \in \langle x_0, \ldots, x_n \rangle \} \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_3).$$

If $[F] \in \text{Sec}_r(X_{3,n})$ is a general point, then

$$F = l_1^3 + \ldots + l_r^3 \quad l_1, \ldots, l_r \in \langle x_0, \ldots, x_n \rangle.$$

$F$ has rank $r$. 

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**Cactus Varieties of Cubic Forms**
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Let \([F] \in Cactus_r(X_{3,n})\) be a general point. Then \(F\) has cactus rank \(r\).

What does \(F\) look like?
Local cactus rank

Definition

The local cactus rank of $F$ is the smallest length of a local scheme in $X_{3,n}$ whose linear span contains $[F]$. 

Theorem

For a general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]^3$ with $n \geq 8$, the cactus rank coincides with the local cactus rank. Thus the cactus rank is computed by a scheme supported at a point $l_3 \in X_{3,n}$ where $l$ is some linear form.
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Thus the cactus rank is computed by a scheme supported at a point $[l^3] \in X_{3,n}$ where $l$ is some linear form.
Let \( l \in \langle x_0, \ldots, x_n \rangle \) and let \( D_l = l^\perp \subset \langle \partial/\partial x_0, \ldots, \partial/\partial x_n \rangle \). Let

\[ L_{F,l} = \langle F, \ l \cdot (D_l(F)), \ l^2 \cdot (D_l^2(F)), \ l^3 \rangle \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_3) \]
A natural local scheme $Z_{F,l} \subset X_{3,n}$

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Denote by

$$Z_{F,l} = L_{F,l} \cap X_{3,n},$$

it is a finite local subscheme supported at $[l^3]$ that spans $L_{F,l}$ and has length equal to

$$\dim_\mathbb{C} \text{Diff}(F(l = 1))$$

the dimension of the space of partials of all orders of the dehomogenization $F(l = 1)$.
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the dimension of the space of partials of all orders of the dehomogenization $F(l = 1)$.

Since $[F] \in L_{F,l}$, the cactus rank and the local cactus rank of $F$ is bounded above by $\dim_{\mathbb{C}} \text{Diff}(F(l = 1))$
For a homogeneous form $G$ of degree $d$, we may similarly define $L_{G,x_0}$ and $Z_{G,x_0} = L_{G,x_0} \cap X_{d,n} \subset X_{d,n} = \mathbb{P}^n$, and consider the image of $Z_{G,x_0}$ in $X_{3,n}$.

If

$$G(x_0 = 1) = g_d + g_{d-1} + \ldots + g_3 + g_2 + g_1 + g_0$$

and $g_i$ is homogeneous of degree $i$, then

$$g_3 + g_2 + g_1 + g_0$$

is the degree 3 tail of $G(x_0 = 1)$. 

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Cactus Varieties of Cubic Forms
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**Proposition**

*If $F(x_0 = 1)$ is the degree 3 tail of $G(x_0 = 1)$, then*

$$[F] \in \langle Z_{G,x_0} \rangle \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_3)$$

*If $Z$ computes the local cactus rank of $F$ and is supported at $[x_0^3]$, then $Z = Z_{G,x_0}$ for some form $G$ such that the degree 3 tail of $G(x_0 = 1)$ coincides with $F(x_0 = 1)$.***
Proposition

For a general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ with $n \geq 8$, the cactus rank is $2n + 2$, and it is computed by $Z_{F,l}$ for any $l$. 

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Proposition

For a general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ with $n \geq 8$ and even cactus rank $c \geq 18$, the cactus rank is computed by $Z_{F,l}$ for some $l$. 
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Proposition

For a general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ with $n \geq 8$ and odd cactus rank $c \geq 17$, the cactus rank is not computed by $Z_{F,l}$ for any $l$. 
A general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ with even local cactus rank $2m, m \leq n$ is projectively equivalent to some

$$f_3 + x_0 f_2 + x_0^2 f_1 + x_0^3 f_0$$

where

$$
\begin{align*}
f_3 & \in \mathbb{C}[x_1, \ldots, x_{m-1}]_3, \\
f_2 & \in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle, \\
f_1 & \in \langle x_1, \ldots, x_n \rangle, \\
f_0 & \in \mathbb{C}.
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$$f_1 \in \langle x_1, \ldots, x_n \rangle,$$
$$f_0 \in \mathbb{C}.$$

The forms of local cactus rank $2n$ form a family of codimension $\binom{n-1}{2} + 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$. 
Corollary

If $n > 2$ and $F$ is a general cubic form with local cactus rank $2m < 2n + 2$.

Then $V(F) \subset \mathbb{P}^n$ is singular along a linear subspace $L$ of dimension $n - m$.

Furthermore there is a hyperplane $H \supset L$ such that $H \cap V(F)$ has multiplicity 3 along $L$. 
A general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$, with odd local cactus rank $2m + 1$, $m \leq n$ is projectively equivalent to some

$$f_3 + x_m x_1^2 + x_0 f_2 + x_0 x_m^2 + x_0^2 f_1 + x_0^3 f_0$$

where

$$f_3 \in \mathbb{C}[x_1, \ldots, x_{m-1}]_3,$$

$$f_2 \in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle,$$

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A general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$, with odd local cactus rank $2m + 1$, $m \leq n$ is projectively equivalent to some

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$$f_0 \in \mathbb{C}.$$

The forms of local cactus rank $2n + 1$, $n > 3$ form a family of codimension $\binom{n-2}{2} - 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$. 
Corollary

If $n > 2$ and $F$ is a general cubic form with odd local cactus rank $2m + 1 < 2n + 2$. Then $V(F) \subset \mathbb{P}^n$ is singular along a linear subspace $L$ of dimension $n - m - 1$.

Furthermore, there is a hyperplane $H \supset L$ such that $H \cap V(F)$ has a tangent cone which is a square along $L$. 
The easy direction of the proof:
If
\[ f = F(x_0 = 1) = f_3 + f_2 + f_1 + f_0 \]
where
\[ f_3 \in \mathbb{C}[x_1, \ldots, x_{m-1}]_3, f_2 \in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle, \]
\[ f_1 \in \langle x_1, \ldots, x_n \rangle, f_0 \in \mathbb{C}, \]
then
\[ \dim_{\mathbb{C}} \text{Diff}(f) = \dim_{\mathbb{C}} \text{Diff}(f_3) = 2m \]
If

\[ f = F(x_0 = 1) = f_3 + x_m x_1^2 + f_2 + x_m^2 + f_1 + f_0 \]

with general

\[ f_3 \in \mathbb{C}[x_1, \ldots, x_{m-1}]_3, \ f_2 \in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle, \]

\[ f_1 \in \langle x_1, \ldots, x_n \rangle, \ f_0 \in \mathbb{C}, \]

then

\[ \dim_{\mathbb{C}} \text{Diff}(f) = \dim_{\mathbb{C}} \text{Diff}(f_3) = 2m + 2. \]
Let
\[ g = \frac{1}{12} x_1^4 + f \]
\[ = \frac{1}{12} x_1^4 + f_3 + x_m x_1^2 + f_2 + x_m^2 + f_1 + f_0, \]
then there exist \( \psi \in \mathbb{C}[\partial/\partial x_1, \ldots, \partial/\partial x_n]_2 \) such that
\[ ((\partial/\partial x_1)^2 - \psi)(g) = (\partial/\partial x_m)(g) = x_1^2 + 2x_m. \]
Therefore
\[ ((\partial/\partial x_1)^2 - (\partial/\partial x_m))g = \psi(g) \in \text{Diff}(x_1^4 + f_3 + f_2 + f_1 + f_0), \]
and
\[ \dim_{\mathbb{C}} \text{Diff}(g) = \dim_{\mathbb{C}} \text{Diff}(x_1^4 + f_3 + f_2 + f_1 + f_0) \]
\[ = \dim_{\mathbb{C}} \text{Diff}(f_3) + 1 = 2m + 1. \]
A simpler example with $n = m = 2$: Let
\[ f = x_1^2 x_2 + x_2^2 \quad \text{and} \quad g = \frac{1}{12} x_1^4 + x_1^2 x_2 + x_2^2 \]
Then
\[ \text{Diff}(f) = \langle f, x_1^2 + 2x_2, 2x_1 x_2, x_1, x_2, 1 \rangle \]
so
\[ \dim_{\mathbb{C}} \text{Diff}(f^3) = 6. \]
A simpler example with $n = m = 2$:
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so
\[ \dim_{\mathbb{C}} \text{Diff}(f_3) = 6. \]
On the other hand,
\[ ((\partial/\partial x_1)^2(g) = (\partial/\partial x_2)(g) = x_1^2 + 2x_2. \]
Therefore
\[ \text{Diff}(g) = \langle g, \frac{1}{3}x_1^3 + 2x_1 x_2, x_1^2 + 2x_2, x_1, 1 \rangle \]
so
\[ \dim_{\mathbb{C}} \text{Diff}(g) = 5. \]
A cubic surface (cubic form in 4 variables) have local cactus rank at most 6 if it has a singular point $p$ and a hyperplane with a triple point at $p$. The family of such forms has codimension 2.

Every cubic surface (cubic form in 4 variables) has local cactus rank 7, since they all have a cuspidal tangent plane section (with a non reduced tangent cone at the singular point).
Casnati, Jelisiejew, Notari (2013):

**Theorem**

*Every local Gorenstein scheme of length* \( r \leq 13 \) *is smoothable.*

Iarrobino (< 1980):

**Theorem**

*There exist nonsmoothable local Gorenstein schemes of length 14.*

Buczyńska, Buczyński (2010):

**Proposition**

*A finite scheme that computes the cactus rank of a form is locally Gorenstein.*
Corollary

When \( r \leq 13 \), then

\[
Cactus_r(X_3, n) = \text{Sec}_r(X_3, n)
\]

When \( r \geq 14 \), then

\[
Cactus_r(X_3, n) \neq \text{Sec}_r(X_3, n)
\]
Theorem

When \( r \leq 17 \), then

\[
\dim \text{Cactus}_r(X_3, n) = \dim \text{Sec}_r(X_3, n) = \min \left\{ rn + r - 1, \binom{n + 3}{3} - 1 \right\}.
\]

When \( 18 \leq r \leq 2n + 2 \), then

\[
\dim \text{Cactus}_r(X_3, n) =
\begin{cases}
(rn + r - 1) + \frac{r(r-2)(r-16)}{48} - 1 & \text{if } r \geq 18 \text{ even}, \\
(rn + r - 1) + \frac{(r-1)(r-3)(r-17)}{48} - 2 & \text{if } r \geq 19 \text{ odd}.
\end{cases}
\]
- If $Z \subset X_{3,n}$ computes the cactus rank of $F$, then it has components

$$Z = Z_1 \cup \ldots \cup Z_s$$

and $F = F_1 + \ldots + F_s$ such that $Z_i$ computes the cactus rank of $F_i$. 

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Cactus Varieties of Cubic Forms
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and $F = F_1 + \ldots + F_s$ such that $Z_i$ computes the cactus rank of $F_i$.

- (Apolarity). Any finite local Gorenstein scheme is isomorphic to $Z_{G,l}$, defined like $Z_{F,l}$ for forms $G$ of any degree.

- Use Iarrobino’s results on Artinian Gorenstein rings to parameterize the family of local Gorenstein schemes of given length.
Problem:

Find an effective algorithm to compute the local cactus rank of a cubic form $F$. 
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Find an effective algorithm to compute the local cactus rank of a cubic form $F$.

Thank you