Extended Abstract

SUBRESULTANTS IN MULTIPLE ROOTS

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1. Introduction

In [DKS2006], we presented Poisson-like formulas for multivariate subresultants in terms of the roots of a system given by a subset of the input polynomials, provided that all these roots were *simple*, i.e. that the ideal generated by the input polynomials is radical and zero-dimensional. Later on, in [DHKS2007, DHKS2009], the formulas were extended for subresultants and double Sylvester sums in the case of univariate polynomials with single roots. As it was pointed out by one of the referees in the previous MEGA'2007, it would be interesting to work out these results for the case of multiple roots. This draft is a first attempt in that direction. We extend the results given in [DKS2006] for the case of multiple roots, and also explore combinatorial formulas for the univariate case for particular cases of multiple roots.

Multivariate resultants were mainly introduced by Macaulay in [Mac1902], after earlier work by Euler, Sylvester and Cayley, while multivariate subresultants were first defined by Gonzalez-Vega in [GLV1990, GLV1991], generalizing Habicht's method [Hab1948]. The notion of subresultants that we use in the present paper was introduced by Chardin in [Cha1995]. It works as follows: let $f_1^h, \ldots, f_s^h$ be a system of generic homogeneous polynomials in $K[x_0, x_1, \ldots, x_n]$ of degrees $d_i = \deg(f_i^h)$ with parametric coefficients, where $s \leq n+1$ and K is the coefficient field of $f_1^h, \ldots, f_s^h$. Let $\mathcal{H}_{d_1,\ldots,d_s} : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of a complete intersection given by s homogeneous polynomials in $n+1$ variables of degrees $d_1, \ldots, d_s$. Fix $t \in \mathbb{N}$ and let $\mathcal{S}$ be a set of $\mathcal{H}_{d_1,\ldots,d_s}(t)$ monomials of degree t. The *subresultant* $\Delta_{\mathcal{S}}$ is a polynomial in K whose degree in the coefficients of f_i^h is $\mathcal{H}_{d_1,\ldots,d_{i-1},d_{i+1},\ldots,d_s}(t-d_i)$ for $i = 1, \ldots, s$, having the following universal property: $\Delta_{\mathcal{S}}$ vanishes at a particular coefficient specialization $\tilde{f}_1^h, \ldots, \tilde{f}_s^h \in \mathbb{C}[x_0, \ldots, x_n]$ if and only if $I_t \cup \mathcal{S}$ does not generate the space of all forms of degree t. Here, I_t is the degree t part of the ideal generated by the $\tilde{f}_i^h$'s (see [Cha1995]).

The constructions in [GLV1990, Cha1995] generalize the classical univariate subresultants in the sense that they provide the coefficients of certain polynomials in I_t, which in the univariate case include the greatest common divisor of two given polynomials.

2. Univariate Case: Poisson-like formulas

2.1. The Generalized Vandermonde and Wronskian Matrices.

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Throughout this abstract we assume K is an algebraically closed field of characteristic zero. We set the notation for slight modifications of a case of generalized Wronskian matrices and of generalized Vandermonde matrices:

Notation 2.1. *Given a polynomial $p(t) \in K[t]$, $E = \{e_1, \dots, e_k\} \subseteq \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$, the following (non-necessarily square) $k \times m$ matrix will be a block of the generalized Wronskian matrix:*

$$W_{p,E}(t,m) := \begin{array}{c} m \\ \boxed{\begin{array}{cccc} (t^{e_1}p) & (t^{e_1}p)' & \dots & \frac{(t^{e_1}p)^{(m-1)}}{(m-1)!} \\ \vdots & \vdots & & \vdots \\ (t^{e_k}p) & (t^{e_k}p)' & \dots & \frac{(t^{e_k}p)^{(m-1)}}{(m-1)!} \end{array}} \end{array} \; k$$

and for $\alpha \in K$ we simply denote $W_{p,E}(\alpha, m) := W_{p,E}(t,m)(\alpha)$, that is, the matrix obtained by specializing every coefficient in α.
Now we let $\alpha_1, \dots, \alpha_r \in K$ be all distinct, and $m_1, \dots, m_r \in \mathbb{N}$, and we set $(\vec{\alpha}, \vec{m}) := (\alpha_1, m_1; \dots; \alpha_r, m_r)$ and $m := m_1 + \cdots + m_r$. We define the Wronskian matrix as

$$W_{p,E}(\vec{\alpha}, \vec{m}) := \begin{array}{c} m \\ \boxed{\begin{array}{c|c|c} W_{p,E}(\alpha_1, m_1) & \dots & W_{p,E}(\alpha_r, m_r) \end{array}} \end{array} \; k \; .$$

When $E = \{0, \dots, k-1\}$, we write $W_{p,k}$ instead of $W_{p,E}$, and, in this case, when $k = m$ we simply write W_p.

In the particular case $p = 1$, we obtain the generalized Vandermonde matrix

$$V_E(\vec{\alpha}, \vec{m}) := W_{1,E}(\vec{\alpha}, \vec{m}).$$

When $E = \{0, \dots, k-1\}$, we write V_k instead of V_E, and, in this case, when $k = m$, we simply write V. For instance

$$V(\alpha, 3; \beta, 2) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ \alpha & 1 & 0 & \beta & 1 \\ \alpha^2 & 2\alpha & 1 & \beta^2 & 2\beta \\ \alpha^3 & 3\alpha^2 & 3\alpha & \beta^3 & 3\beta^2 \\ \alpha^4 & 4\alpha^3 & 6\alpha^2 & \beta^4 & 4\beta^3 \end{array} \right]$$

With these notations, we have the following classical first statement (see for instance [Ait1939]), while the second is easily obtainable performing row operations in the case of one block, and by induction in the size of the matrix in general:

$$\begin{aligned} \det\big(V(\vec{\alpha}, \vec{m})\big) &= \prod_{1 \leq i < j \leq r} (\alpha_j - \alpha_i)^{m_i m_j}, \\ \det\big(W_p(\vec{\alpha}, \vec{m})\big) &= \Big(\prod_{1 \leq i < j \leq r} (\alpha_j - \alpha_i)^{m_i m_j} \Big) p(\alpha_1)^{m_1} \cdots p(\alpha_r)^{m_r}. \end{aligned}$$

Remark 2.2. In the same way that the usual Vandermonde matrix is strongly related to the Lagrange Interpolation Problem, the generalized

Vandermonde matrix is known to be strongly related to the Hermite Interpolation Problem (see for instance [KTO1997]):
Given $\{y_{ij_i}, 1 \le i \le r, 0 \le j_i < m_i\} \subset K$, there exists a unique polynomial p of degree strictly bounded by m which satisfies the following conditions:

$$p^{(j_1)}(\alpha_1) = j_1! y_{1j_1},\ 0 \le j_1 < m_1; \quad \dots \quad ; p^{(j_r)}(\alpha_r) = j_r! y_{rj_r},\ 0 \le j_r < m_r,$$

and this polynomial p is given by the formula

$$p(x) = -\frac{\det \left[\begin{array}{cccc|c} & & & & 1 \\ & & & & x \\ & V(\vec{\alpha}, \vec{m}) & & & \vdots \\ & & & & x^{m-1} \\ \hline y_{10} & y_{11} & \cdots & y_{r(m_r-1)} & 0 \end{array}\right]}{\det\big(V(\vec{\alpha}, \vec{m})\big)} .$$

2.2. **Subresultants in Multiple Roots.**
For $\alpha_1, \dots, \alpha_r \in K$ different and $m_1, \dots, m_r \in \mathbb{N}$, with $m := m_1 + \cdots + m_r$, we set

$$f := a_m x^m + \cdots + a_0 = (x - \alpha_1)^{m_1} \cdots (x - \alpha_r)^{m_r} \ \in\ K[x]$$

and $g(x) = b_n x^n + \cdots + b_0 \in K[x]$, monic polynomials of degrees m and n respectively. Our aim is to give an expression for the d-th subresultant

$$Sres_d(f, g) := \det \begin{array}{c} \overbrace{\hphantom{a_m \quad \cdots \quad \cdots \quad a_{d+1-(n-d-1)}}}^{m+n-2d} \\ \left|\begin{array}{ccccc} a_m & \cdots & & \cdots & a_{d+1-(n-d-1)} \quad x^{n-d-1} f(x) \\ & \ddots & & \vdots & \vdots \\ & & a_m & \cdots \quad a_{d+1} & f(x) \\ \hline b_n & \cdots & & \cdots & b_{d+1-(m-d-1)} \quad x^{m-d-1} g(x) \\ & \ddots & & \vdots & \vdots \\ & & b_n & \cdots \quad b_{d+1} & g(x) \end{array}\right| \end{array} \begin{array}{l} \\ n-d \\ \\ \\ m-d \\ \end{array},$$

where $d \in \mathbb{Z}$, $0 \le d \le \min\{m, n\}$, in terms of the data $\alpha_1, \dots, \alpha_r$, $m_1, \dots, m_r$ and g.

The following claim is a generalization of [Hon1999, Th. 3.1] and [DHKS2007, Lem. 2] for the case when f has multiple roots:

Theorem 2.3.

$$Sres_d(f, g) = \frac{\det \left[\begin{array}{c} W_{x-t,d}(\vec{\alpha}, \vec{m}) \\ \hline W_{g,m-d}(\vec{\alpha}, \vec{m}) \end{array}\right]}{\det\big(V(\vec{\alpha}, \vec{m})\big)} = \frac{\det \left[\begin{array}{c|c} & 1 \\ & x \\ V_{d+1}(\vec{\alpha}, \vec{m}) & \vdots \\ & x^d \\ \hline & 0 \\ W_{g,m-d}(\vec{\alpha}, \vec{m}) & \vdots \\ & 0 \end{array}\right]}{\det\big(V(\vec{\alpha}, \vec{m})\big)}.$$

Sketch of the Proof. Let $f = \sum_{i=0}^{m} a_i x^i$ and $g = \sum_{i=0}^{n} b_i x_i$. As in [DHKS2007], we define the following matrices

$$M_f := \begin{array}{c} {\scriptstyle m+n-d} \\ \left[\begin{array}{ccccc} a_0 & \dots & a_m & & \\ & \ddots & & \ddots & \\ & & a_0 & \dots & a_m \end{array}\right] \end{array} {\scriptstyle n-d}\ , \qquad M_g := \begin{array}{c} {\scriptstyle m+n-d} \\ \left[\begin{array}{ccccc} b_0 & \dots & b_n & & \\ & \ddots & & \ddots & \\ & & b_0 & \dots & b_n \end{array}\right] \end{array} {\scriptstyle m-d}\ .$$

and

$$S_d := \begin{array}{c} {\scriptstyle m+n-d} \\ \left[\begin{array}{c} M_{x-t} \\ \hline M_f \\ \hline M_g \end{array}\right] \end{array} \begin{array}{c} {\scriptstyle d} \\ {\scriptstyle n-d} \\ {\scriptstyle m-d} \end{array} \quad \text{where} \quad M_{x-t} := \begin{array}{c} {\scriptstyle m+n-d} \\ \left[\begin{array}{ccccccc} x & -1 & 0 & \dots & & \dots & 0 \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & x & -1 & 0 & \dots & 0 \end{array}\right] \end{array} {\scriptstyle d}\ .$$

We have ([DHKS2007, Lem. 1]):

$$Sres_d(f, g) = (-1)^{(n-d)(m-d)} \det(S_d).$$

The first equality of the statement is obtained taking determinants in the following product of matrices

$$\begin{array}{c} {\scriptstyle d} \\ {\scriptstyle n-d} \\ {\scriptstyle m-d} \end{array} \begin{array}{c} {\scriptstyle m+n-d} \\ \left[\begin{array}{c} M_{x-t} \\ \hline M_f \\ \hline M_g \end{array}\right] \end{array} \begin{array}{cc} {\scriptstyle m} & {\scriptstyle n-d} \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} & 0 \\ V_{m+n-d}(\vec{\alpha}, \vec{m}) & \\ & Id_{n-d} \end{array}\right]} \end{array} \begin{array}{c} {\scriptstyle m} \\ \\ {\scriptstyle n-d} \end{array} = \begin{array}{cc} {\scriptstyle m} & {\scriptstyle n-d} \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} W_{x-t,d}(\vec{\alpha}, \vec{m}) & * \\ \hline \mathbf{0} & M'_f \\ \hline W_{g,m-d}(\vec{\alpha}, \vec{m}) & * \end{array}\right]} \end{array} \begin{array}{c} {\scriptstyle d} \\ {\scriptstyle n-d} \\ {\scriptstyle m-d} \end{array},$$

because $M_{x-t} \cdot V_{m+n-d} = W_{x-t,d}$, $M_f \cdot V_{m+n-d} = W_{f,n-d} = [0]$ since $f(\alpha_i) = \cdots = f^{(m_i-1)}(\alpha_i) = 0$, and $M_g \cdot V_{m+n-d} = W_{g,m-d}$. Finally M'_f is a lower triangular matrix with diagonal entries $a_m = 1$.

The second equality is easily obtained from the first one by performing row operations in the matrix in the numerator.

□

We easily get from Theorem 2.3 an expression for the following particular case of d:

Observation 2.4. For $d = m-1 < n$, Remark 2.2 shows that $Sres_{m-1}(f, g)$ coincides, up to a sign, with the Hermite interpolator polynomial p associated to the m conditions

$$p^{(j_i)}(\alpha_i) = g^{(j_i)}(\alpha_i),\ 1 \le i \le r, 0 \le j_i < m_i,$$

since the coefficients of $W_{g,1}$ are $g^{(j_i)}(\alpha_i)/j_i!$. This is the generalization of the well-known fact that when f has simple roots, $Sres_{m-1}(f, g)$ is, up to a sign, the Lagrange interpolator polynomial m which coincides with g in the m values $\alpha_1, \dots, \alpha_m$.

3. Univariate Case: some formulas in roots

3.1. The case $d = 1$.
In case $d = 1$ we can get formulas for the subresultant for the extremal cases $f = (x-\alpha_1)\cdots(x-\alpha_m)$, $\alpha_i \neq \alpha_j$ for $i \neq j$ (all simple roots) and $f = (x-\alpha)^m$ (one single multiple root), and $g = (x-\beta_1)^{n_1}\cdots(x-\beta_s)^{n_s}$ arbitrary. However an expression for the general case $f = (x-\alpha_1)^{m_1}\cdots(x-\alpha_r)^{m_r}$ is still lacking.
The case when f has all simple roots is a direct consequence of Theorem 2.3, more precisely of the previous versions [Hon1999, Th. 3.1] and [DHKS2007, Lem. 2].

Observation 3.1. Let $f = (x-\alpha_1)\cdots(x-\alpha_m)$, $\alpha_i \neq \alpha_j$ for $i \neq j$ and g be an arbitrary monic polynomial. Then

$$Sres_1(f,g) = \sum_{i=1}^{m}\Big(\prod_{j\neq i}\frac{g(\alpha_j)}{\alpha_j-\alpha_i}\Big)(x-\alpha_i).$$

The case when f has one single multiple root has a much more difficult expression and proof:

Proposition 3.2. *Let $f = (x-\alpha)^m$ and let $g = (x-\beta_1)^{n_1}\cdots(x-\beta_s)^{n_s}$. Then*

$$\begin{aligned} Sres_1(f,g) \;\; = \;\; & \sum_{\ell_1+\cdots+\ell_s=m-2}\Big(\prod_{j=1}^{s}\binom{n_j-1+\ell_j}{n_j-1}(\alpha-\beta_j)^{(m-1)n_j-\ell_j}\Big)+ \\ & + \sum_{k_1+\cdots+k_s=m-1}\Big(\prod_{j=1}^{s}\binom{n_j-1+k_j}{n_j-1}(\alpha-\beta_j)^{(m-1)n_j-k_j}\Big)(x-\alpha), \end{aligned}$$

where by convention $\binom{k}{j} = 0$ if $j > k$.

Sketch of proof. In this case, since $\det(V(\alpha,m)) = 1$, we have that

$$Sres_1(f,g) = \det\begin{bmatrix} W_{x-t,1}(\alpha,m) \\ \hline W_{g,m-1}(\alpha,m)\end{bmatrix} = \det\begin{array}{c} \left[\begin{array}{cccc} (x-\alpha) & -1 & & \\ \hline g(\alpha) & \dots & \dots & \frac{g^{(m-1)}(\alpha)}{(m-1)!} \\ & \ddots & & \vdots \\ 0 & & g(\alpha) & g'(\alpha) \end{array}\right] \end{array}\begin{array}{l} 1 \\ \\ m-1 \\ \\ \end{array},$$

where the second identity is obtained after performing row operations. Setting $D_0 := 1$ and for $k \geq 1$,

$$D_k = \det\begin{array}{c} k \\ \left[\begin{array}{cccc} g' & \frac{g''}{2!} & \dots & \frac{g^{(k)}}{k!} \\ g & g' & \dots & \frac{g^{(k-1)}}{(k-1)!} \\ & \ddots & \ddots & \\ 0 & & g & g' \end{array}\right] \end{array}\begin{array}{l} \\ \\ k \\ \end{array},$$

we deduce that $Sres_1(f,g) = (x-\alpha)D_{m-1}(\alpha) + g(\alpha)D_{m-2}(\alpha)$. We finally prove by induction, developing D_k by the last column, that for $g = (x -$

$\beta_1)^{n_1}\cdots(x-\beta_s)^{n_s}$,

$$D_k(\alpha)=\sum_{k_1+\cdots+k_s=k}\binom{n_1-1+k_1}{n_1-1}\cdots\binom{n_s-1+k_s}{n_s-1}(\alpha-\beta_1)^{kn_1-k_1}\cdots(\alpha-\beta_s)^{kn_s-k_s}.$$

Here we use the identities

$$\frac{g^{(i)}(x)}{i!}=\sum_{i_1+\cdots+i_s=i}\binom{n_1}{i_1}\cdots\binom{n_s}{i_s}(x-\beta_1)^{n_1-i_1}\cdots(x-\beta_s)^{n_s-i_s},\quad 1\le i\le k,$$

and

$$\sum_{\substack{i_1\le k_1,\dots,i_s\le k_s\\ i_1+\cdots+i_s\neq 0}}(-1)^{i_1+\cdots+i_s-1}\prod_{j=1}^{s}\binom{n_j}{i_j}\binom{n_j-1+(k_j-i_j)}{n_j-1}=\prod_{j=1}^{s}\binom{n_j+k_j-1}{n_j-1},$$

a multivariate version of [GKP2000, Id. 5.24], that can be easily derived by induction in the number of factors. □

Proposition 3.2 seems to suggest that the combinatorial formulas for the subresultant are really complicated.

3.2. **Sylvester's double sums like formulas.**

Next we describe some results we could get in terms of the roots of both polynomials in the general case when $f=(x-\alpha_1)^{m_1}\cdots(x-\alpha_r)^{m_r}$ and $g(x)=(x-\beta_1)^{n_1}\cdots(x-\beta_s)^{n_s}$ with $m=\sum_{i=1}^r m_i$ and $n:=\sum_{i=1}^s n_i$. The following is the analogue of Lemma 3 in [DHKS2007] for the multiple roots case.

Proposition 3.3. *Let $P,Q\subset\{0,1,\dots,d-1\}$ be ordered sets such that $P\cup Q=\{0,1,\dots,d-1\}$ and $P\cap Q=\emptyset$. Let $|P|=p$, $|Q|=q$.*

$$Sres_d(f,g)=\pm\frac{\det\begin{array}{cc} m & n \\ \left|\begin{array}{c|c} W_{x-t,P}(\vec{\alpha},\vec{m}) & 0 \\ \hline 0 & W_{x-t,Q}(\vec{\beta},\vec{n}) \\ \hline V_{m+n-d}(\vec{\alpha},\vec{m}) & V_{m+n-d}(\vec{\beta},\vec{n})) \end{array}\right| \end{array}\begin{array}{l} \\ p\\ q\\ m+n-d\end{array}}{\det\big(V(\vec{\alpha},\vec{m})\big)\ \det\big(V(\vec{\beta},\vec{n})\big)}\ .$$

Sketch of proof. From Theorem 2.3 we have that

$$Sres_d(f,g)\ \det\big(V(\vec{\alpha},\vec{m})\big)\ \det\big(V(\vec{\beta},\vec{n})\big)\ =\ \pm\det\begin{array}{cc} m & n \\ \left|\begin{array}{c|c} W_{x-t,d}(\vec{\alpha},\vec{m}) & 0 \\ \hline V_n(\vec{\alpha},\vec{m}) & V_n(\vec{\beta},\vec{n}) \\ \hline W_{g,m-d}(\vec{\alpha},\vec{m}) & 0 \end{array}\right| \end{array}\begin{array}{l} \\ d\\ n\\ m-d\end{array}$$

$$=\ \pm\det\begin{array}{l} \\ d\\ n\\ m-d\end{array}\begin{array}{ccc} d & n & m-d \\ \left|\begin{array}{c|c|c} Id & 0 & 0 \\ \hline 0 & Id & 0 \\ \hline 0 & \multicolumn{2}{c}{M_g} \end{array}\right| \end{array}\ \begin{array}{cc} m & n \\ \left|\begin{array}{c|c} W_{x-t,d}(\vec{\alpha},\vec{m}) & 0 \\ \hline V_{m+n-d}(\vec{\alpha},\vec{m}) & V_{m+n-d}(\vec{\beta},\vec{n}) \end{array}\right| \end{array}\ .$$

Note that the first matrix is lower triangular with diagonal entries 1, so its determinant is 1. As $m+n-d>d$, the obvious subtractions and permutations of rows yield that the the determinant of the second matrix in the previous product equals

$$\pm \det \begin{array}{|c|c|l} \multicolumn{1}{c}{m} & \multicolumn{1}{c}{n} & \\ \hline W_{x-t,P}(\vec{\alpha},\vec{m}) & 0 & p \\ \hline 0 & -W_{x-t,Q}(\vec{\beta},\vec{n}) & q \\ \hline V_{m+n-d}(\vec{\alpha},\vec{m}) & V_{m+n-d}(\vec{\beta},\vec{n}) & m+n-d \\ \hline \end{array} .$$

□

Remark 3.4. Note that expanding the determinant in the numerator of Proposition 3.3 by square blocks of the first p rows and square blocks of the second q rows, and then taking the sum over all possible partitions $P, Q \subset \{0, \dots, d-1\}$ yields that

$$\binom{d}{p} Sres_d(f,g) \det\big(V(\vec{\alpha},\vec{m})\big) \det\big(V(\vec{\beta},\vec{n})\big) =$$

$$\pm \sum_{\substack{X\subset\{1,\dots,m\}\\ Y\subset\{1,\dots,n\}\\ |X|=p,|Y|=q}} \operatorname{sg}(X)\operatorname{sg}(Y) \det\big(W_{x-t,p,X}(\vec{\alpha},\vec{m})\big) \det\big(W_{x-t,q,Y}(\vec{\beta},\vec{n})\big)\cdot$$

$$\cdot \det\big(V_{m-p,\bar{X}}(\vec{\alpha},\vec{m})\big) \det\big(V_{n-q,\bar{Y}}(\vec{\beta},\vec{n})\big),$$

which is a weaker version of the Sylvester double sum formula described in [DHKS2007]. Here $\operatorname{sg}(X) = (-1)^{\sigma(X)}$ denotes the number of transpositions needed to take $X \cup \big((1,\dots,m)\setminus X\big)$ to $(1,\dots,m)$, $W_{x-t,p,X}$ denotes the submatrix of $W_{x-t,p}$ with columns corresponding to X and $V_{n-p,\bar{X}}$ denotes the submatrix of V_{n-p} with columns corresponding to the complement of X; and similarly for $\operatorname{sg}(Y)$, $W_{x-t,q,Y}$ and $V_{n-q,\bar{Y}}$.

However, in the case of multiple roots, submatrices of generalized Vandermonde matrices are not always generalized Vandermonde matrices, so in general their determinants cannot be expressed as products of differences. That is why we could only obtain a weaker version of the Sylvester double sum formula.

4. Multivariate Case: Poisson-like formulas

Now we turn to the multivariate case, and our goal is to generalize Theorem 3.2 in [DKS2006] to the case of multiple roots. We briefly recall here this statement, and refer the reader to the article for more background on this topic.

For $n \in \mathbb{N}$ and $1 \le i \le n+1$, let

$$f_i := \sum_{|\alpha|\le d_i} a_{i\alpha}\mathbf{x}^\alpha \;\in\; K[\mathbf{x}],$$

where $\alpha = (\alpha_1,\dots,\alpha_n) \in (\mathbb{Z}_{\ge 0})^n$, $\mathbf{x}^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1+\cdots+\alpha_n$, and K a field of characteristic zero.

Fix $t \in \mathbb{N}$. Let $k := \mathcal{H}_{d_1\dots d_{n+1}}(t)$ be the Hilbert function at t of a regular sequence of $n+1$ homogeneous polynomials in $n+1$ variables of degrees $d_1,\dots,d_{n+1}$, i.e. $k = \#\{\mathbf{x}^\alpha : |\alpha| \le t, \alpha_i < d_i, 1 \le i \le n, t-|\alpha| < d_{n+1}\}$.

We set

$$\mathcal{S} := \{\mathbf{x}^{\gamma_1},\dots,\mathbf{x}^{\gamma_k}\} \subset K[\mathbf{x}]_t$$

a set of k monomials of degree bounded by t, and

$$\Delta_{\mathcal{S}} := \Delta^{(t)}_{\mathcal{S}^h}(f_1^h, \dots, f_{n+1}^h),$$

the *order* t *subresultant of* $f_1^h, \dots, f_{n+1}^h$ *with respect to* $\mathcal{S}^h := \{\mathbf{x}^{\gamma_1} x_{n+1}^{t-|\gamma_1|}, \dots, \mathbf{x}^{\gamma_k} x_{n+1}^{t-|\gamma_k|}\}$. (Here, f_i^h denotes the homogenization of f_i by the variable x_{n+1}.)

We set $\rho := (d_1 - 1) + \cdots + (d_n - 1)$ and for $j \geq 0$, $\tau_j := \mathcal{H}_{d_1 \dots d_n}(j)$, the Hilbert function at j of a regular sequence of n homogeneous polynomials in n variables of degrees $d_1, \dots, d_n$. Also

$$(1)\quad \mathcal{T}_j := \begin{cases} \textit{any} \text{ set of } \tau_j \text{ monomials of degree } j & \text{if } j \geq \max\{0, t - d_{n+1} + 1\}, \\ \{\mathbf{x}^\alpha : |\alpha| = j, \alpha_i < d_i \text{ for } 1 \leq i \leq n\} & \text{if } 0 \leq j < t - d_{n+1} + 1. \end{cases}$$

We denote $\mathcal{T} := \cup_{j \geq 0} \mathcal{T}_j$ and $\mathcal{T}^* := \cup_{j=t+1}^{\rho} \mathcal{T}_j$. Note that $\#\mathcal{T} = \mathbf{d}$, where $\mathbf{d} := d_1 \cdots d_n$ is the *Bézout number*, the number of common solutions of $f_1, \dots, f_n$ in $\overline{K}^n$. Let $\mathcal{T} := \{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_{\mathbf{d}}}\}$ and assume that for $s := |\mathcal{T}^*|$ we have $\mathcal{T}^* = \{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_s}\}$. We also set

$$(2)\qquad \mathcal{R} := \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_r}\} = \{\mathbf{x}^\alpha, |\alpha| \leq t, \alpha_j < d_j \text{ for } j < i \text{ and } \alpha_i \geq d_i\}.$$

Finally, for $1 \leq i \leq n$, let $\overline{f}_i$ be the homogeneous component of degree d_i of f_i, and $\overline{\Delta}_{\mathcal{T}_j} := \Delta^{(j)}_{\mathcal{T}_j}(\overline{f}_1, \dots, \overline{f}_n)$ be the order j subresultant of $\overline{f}_1, \dots, \overline{f}_n$ with respect to $\mathcal{T}_j$.

4.1. **Poisson-like formula for subresultants: the generic case.** Let $\{\xi_1, \dots, \xi_{\mathbf{d}}\}$ be the set of all common roots of $f_1, \dots, f_n$ in $\overline{K}^n$ *which we for now assume they are all simple*, and $\mathcal{V}_{\mathcal{T}} := \det(\xi_j^{\alpha_i})_{1 \leq i,j \leq \mathbf{d}}$, the generalized Vandermonde determinant associated to $\mathcal{T}$.

In [DKS2006] we defined

$$(3)\qquad \mathcal{O}_{\mathcal{S}} := \begin{array}{|ccc|c} \xi_1^{\gamma_1} & \cdots & \xi_{\mathbf{d}}^{\gamma_1} & \\ \vdots & & \vdots & k \\ \xi_1^{\gamma_k} & \cdots & \xi_{\mathbf{d}}^{\gamma_k} & \\ \hline \xi_1^{\alpha_1} & \cdots & \xi_{\mathbf{d}}^{\alpha_1} & \\ \vdots & & \vdots & s \\ \xi_1^{\alpha_s} & \cdots & \xi_{\mathbf{d}}^{\alpha_s} & \\ \hline \xi_1^{\beta_1} f_{n+1}(\xi_1) & \cdots & \xi_{\mathbf{d}}^{\beta_1} f_{n+1}(\xi_{\mathbf{d}}) & \\ \vdots & & \vdots & r \\ \xi_1^{\beta_r} f_{n+1}(\xi_1) & \cdots & \xi_{\mathbf{d}}^{\beta_r} f_{n+1}(\xi_{\mathbf{d}}) & \end{array} \in \overline{K}^{\mathbf{d} \times \mathbf{d}},$$

and

$$V_{\mathcal{T}} := \begin{vmatrix} \xi_1^{\alpha_1} & \cdots & \xi_{\mathbf{d}}^{\alpha_1} \\ \vdots & & \vdots \\ \xi_1^{\alpha_{\mathbf{d}}} & \cdots & \xi_{\mathbf{d}}^{\alpha_{\mathbf{d}}} \end{vmatrix} \in \overline{K}^{\mathbf{d} \times \mathbf{d}}.$$

Theorem 4.1. [DKS2006, Th. 3.2] *For any* $t \in \mathbb{Z}_{\geq 0}$ *and for any* $\mathcal{S} = \{\mathbf{x}^{\gamma_1}, \dots, \mathbf{x}^{\gamma_k}\} \subset K[\mathbf{x}]_t$ *of cardinality* $k = \mathcal{H}_{d_1 \dots d_{n+1}}(t)$*, the order* t *subresultant* $\Delta_{\mathcal{S}}$ *satisfies:*

$$\Delta_{\mathcal{S}} = \pm \left(\prod_{j=t-d_{n+1}+1}^{t} \overline{\Delta}_{\mathcal{T}_j} \right) \frac{\det \mathcal{O}_{\mathcal{S}}}{\det V_{\mathcal{T}}}.$$

4.2. **Poisson-like formula for subresultants: the multiple roots case.**

In the multivariate case the multiplicity structure of roots of a system can be described by the vanishing of certain partial derivatives of the defining polynomials in the given roots. A theory of the multiplicity structure in the language of dual algebras can be found in [MMM1995]. Here we adopt the notation in [MR2002].

Notation 4.2.

- *We use Λ or $\Lambda(\partial)$ to denote a polynomial in the partial derivatives ∂_{x_i}, i.e. $\Lambda \in K[\partial_{x_1} \dots \partial_{x_n}]$.*
- *For $\Lambda \in K[\partial_{x_1} \dots \partial_{x_n}]$, $p \in \mathbb{C}[\mathbf{x}]$ and $\xi \in \overline{K}^n$, we use $\Lambda(p)(\xi)$ to denote Λ applied to the polynomial p, and the result evaluated at ξ.*

In this section we assume that the polynomials $f_1, \dots, f_n$ have common roots $\xi_1, \dots, \xi_m \in \overline{K}$, and their multiplicity structure is given by $\mathbf{\Lambda}_i = \{\Lambda_{i,1}, \dots, \Lambda_{i,l_i}\} \subset K[\partial_{x_1} \dots \partial_{x_n}]$ for $i = 1, \dots, m$. Furthermore, we assume that $\mathbf{d} = \sum_{i=1}^{m} l_i$, i.e. there are no roots at infinity.
The following definition gives a (slight modification of the) multivariate analogue of Definition 2.1:

Definition 4.3. *Given*

- *Support $E = \{\alpha_1, \dots, \alpha_t\} \subset \mathbb{N}^n$;*
- *$\vec{\xi} = (\xi_1, \dots, \xi_m)$;*
- *The* tangential conditions *$\vec{\mathbf{\Lambda}} = (\mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_m)$, where each $\mathbf{\Lambda}_i = \{\Lambda_{i,1}, \dots, \Lambda_{i,l_i}\} \subset K[\partial_{x_1} \dots \partial_{x_n}]$ has cardinality l_i.*

The generalized Vandermonde matrix *corresponding to E and $\mathbf{\Lambda}$ is the following $d \times t$ matrix:*

$$V_E(\vec{\xi}, \vec{\mathbf{\Lambda}}) := \left|\begin{array}{ccccc} \Lambda_{1,1}(\mathbf{x}^{\alpha_1})(\xi_1) & \cdots & \Lambda_{1,l_1}(\mathbf{x}^{\alpha_1})(\xi_1) & \cdots & \Lambda_{m,l_m}(\mathbf{x}^{\alpha_1})(\xi_m) \\ \vdots & & \vdots & & \vdots \\ \Lambda_{1,1}(\mathbf{x}^{\alpha_t})(\xi_1) & \cdots & \Lambda_{1,l_1}(\mathbf{x}^{\alpha_t})(\xi_1) & \cdots & \Lambda_{m,l_m}(\mathbf{x}^{\alpha_t})(\xi_m) \end{array}\right|.$$

Also, we define the generalized Wronskian matrix *for any $p \in K[\mathbf{x}]$ by*

$$W_{p,E}(\vec{\xi}, \vec{\mathbf{\Lambda}}) := \left|\begin{array}{ccccc} \Lambda_{1,1}(\mathbf{x}^{\alpha_1}p)(\xi_1) & \cdots & \Lambda_{1,l_1}(\mathbf{x}^{\alpha_1}p)(\xi_1) & \cdots & \Lambda_{m,l_m}(\mathbf{x}^{\alpha_1}p)(\xi_m) \\ \vdots & & \vdots & & \vdots \\ \Lambda_{1,1}(\mathbf{x}^{\alpha_t}p)(\xi_1) & \cdots & \Lambda_{1,l_1}(\mathbf{x}^{\alpha_t}p)(\xi_1) & \cdots & \Lambda_{m,l_m}(\mathbf{x}^{\alpha_t}p)(\xi_m) \end{array}\right|.$$

We modify the definition of the matrix $\mathcal{O}_{\mathcal{S}}$ defined in (3) as follows:

Definition 4.4. *Let $\mathcal{S} = \{\mathbf{x}^{\gamma_1}, \dots, \mathbf{x}^{\gamma_k}\} \subset K[\mathbf{x}]_t$ of cardinality $k = \mathcal{H}_{d_1 \dots d_{n+1}}(t)$, $\mathcal{T}^* := \cup_{j=t+1}^{\rho} \mathcal{T}_j$ as in (1), and $\mathcal{R}$ as in (2). Let $\vec{\xi} = (\xi_1, \dots, \xi_m)$ and the tangential conditions $\vec{\mathbf{\Lambda}} = (\mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_m)$ be as above. Then*

$$\mathcal{O}_{\mathcal{S}}(\vec{\xi}, \vec{\mathbf{\Lambda}}) := \begin{array}{|c|} \hline V_{\mathcal{S}}\left(\vec{\xi}, \vec{\mathbf{\Lambda}}\right) \\ \hline V_{\mathcal{T}^*}\left(\vec{\xi}, \vec{\mathbf{\Lambda}}\right) \\ \hline W_{f_{n+1},\mathcal{R}}\left(\vec{\xi}, \vec{\mathbf{\Lambda}}\right) \\ \hline \end{array}.$$

The following is the extension of Theorem 4.1 to the multiple roots case.

Theorem 4.5. *For any $t \in \mathbb{Z}_{\geq 0}$ and for any $\mathcal{S} = \{\mathbf{x}^{\gamma_1}, \ldots, \mathbf{x}^{\gamma_k}\} \subset K[\mathbf{x}]_t$ of cardinality $k = \mathcal{H}_{d_1 \ldots d_{n+1}}(t)$, the order t subresultant $\Delta_{\mathcal{S}}$ satisfies:*

$$\Delta_{\mathcal{S}} = \pm \left(\prod_{j=t-d_{n+1}+1}^{t} \overline{\Delta}_{\mathcal{T}_j} \right) \frac{\det \mathcal{O}_{\mathcal{S}}(\vec{\xi}, \vec{\mathbf{\Lambda}})}{\det V_{\mathcal{T}}(\vec{\xi}, \vec{\mathbf{\Lambda}})}.$$

Sketch of the Proof. Follow the proof of Theorem 3.2 in [DKS2006] with care, all the statements follow straightforwardly. □

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