Alastair Hall **ECG 790: Advanced Econometrics** Fall 2006

Handout on the Weak Law of Large Numbers

In this handout we present a proof of the Weak Law of Large Numbers and use this result to highlight important differences between the independent sampling (cross section or panel data) setting and time series cases.

The proof of the WLLN rests on two intermediate results: Markov's Inequality and Chebyshev's Inequality. For completeness, we state and prove these two results first. Once this has been done, we consider the proof of the WLLN in the iid case and then its extension to the covariance stationary case.

Lemma 1 Markov's inequality
If w is a non-negative random variable and δ is a positive constant then

$$P[\, w \,\geq\, \delta \,] \;\leq\; \frac{E[w]}{\delta}$$

Proof: From the definition of the expectations operator, it follows that

$$E[w] \;=\; E[w \,|\, w \,\geq\, \delta]P[w \,\geq\, \delta] \,+\, E[w \,|\, w \,<\, \delta]P[w \,<\, \delta] \tag{1}$$

Since all the terms on the right hand side of (1) are non-negative, it follows from this equation that

$$E[w] \;\geq\; E[\, w \,|\, w \,\geq\, \delta\,]P[w \,\geq\, \delta] \tag{2}$$

Now $E[\, w \,|\, w \,\geq\, \delta\,] \geq \delta$ and so (3) implies that

$$E[w] \;\geq\; \delta\, P[w \geq \delta] \tag{3}$$

The desired result is obtained by dividing both sides of (3) by δ. $\diamond$

Lemma 2 Chebyshev's Inequality
If u is a random variable and c and ϵ are constants with $\epsilon > 0$ then

$$P[\, |u - c| \,\geq\, \epsilon\,] \;\leq\; \frac{E[\,(u \,-\, c)^2\,]}{\epsilon^2}$$

Proof: Since $|u-c| \geq \epsilon$ and $(u-c)^2 \geq \epsilon^2$ are the same events, it follows that

$$P[\,|u-c| \,\geq\, \epsilon\,] \;=\; P[\,(u-c)^2 \,\geq\, \epsilon^2\,] \tag{4}$$

Note that $(u-c)^2$ is a non-negative random variable. The desired result then follows by applying Markov's inequality with $w = (u-c)^2$ and $\delta = \epsilon^2$ and then using (4). $\diamond$

Theorem 1 Weak Law of Large Numbers for iid variables
If $\{v_t;\, t = 1, 2, \ldots T\}$ be a sequence of iid random variables with $E[v_t] = \mu$, $Var[v_t] = \sigma^2$ and $E[v_t^2] < \infty$ then $T^{-1}\sum_{t=1}^T v_t \xrightarrow{p} \mu$.

Proof: Define $\bar{v}_T = T^{-1}\sum_{t=1}^T v_t$. From the definition of convergence in probability, we need to show

$$\lim_{T\to\infty} P[|\bar{v}_T \,-\, \mu| \,<\, \epsilon] \;=\; 1$$

for any positive constant ϵ, or equivalently

$$\lim_{T\to\infty} P[|\bar{v}_T \,-\, \mu| \,\geq\, \epsilon] \;=\; 0 \tag{5}$$

To this end, we consider $P[|\bar{v}_T \,-\, \mu| \,\geq\, \epsilon]$. Using Chebyshev's inequality with $u = \bar{v}_T$ and $c = \mu$, it follows that

$$P[|\bar{v}_T - \mu| \,\geq\, \epsilon] \;\leq\; \frac{E[(\bar{v}_T - \mu)^2]}{\epsilon^2} \tag{6}$$

Now $\bar{v}_T - \mu = T^{-1}\sum_{t=1}^T (v_t - \mu)$ and so

$$E[(\bar{v}_T - \mu)^2] \;=\; E[\{T^{-1}\sum_{t=1}^T (v_t - \mu)\}^2] \tag{7}$$

$$= \; E[T^{-2}\sum_{t=1}^T \sum_{s=1}^T (v_t - \mu)(v_s - \mu)] \tag{8}$$

$$= \; T^{-2}\sum_{t=1}^T \sum_{s=1}^T E[(v_t - \mu)(v_s - \mu)] \tag{9}$$

Since $\{v_t;\, t = 1, 2, \ldots T\}$ is an iid sequence,

$$E[(v_t - \mu)(v_s - \mu)] \;=\; \sigma^2, \text{ for } t = s \tag{10}$$

$$= \; 0, \text{ for } t \neq s \tag{11}$$

Using (7)-(11), equation (6) becomes

$$P[|\bar{v}_T - \mu| \,\geq\, \epsilon] \;\leq\; \frac{\sigma^2}{T\epsilon^2} \tag{12}$$

Taking limits on both sides of (12) yields

$$\lim_{T\to\infty} P[|\bar{v}_T - \mu| \ \geq \ \epsilon] \ \leq \ \lim_{T\to\infty} \frac{\sigma^2}{T\epsilon^2} \tag{13}$$

Since $\sigma^2 < \infty$ it follows that the right hand side of (13) is zero for any $\epsilon \neq 0$. Therefore, it follows from (13) that (5) holds and $\bar{v}_T \xrightarrow{p} \mu$ which is the desired result. $\diamond$.

A review of the proof indicates that a key condition for the result is the finiteness of $E[T(\bar{v}_T - \mu)^2]$. In the setting above, this result follows from the twin assumptions that the random variables have finite variance and are mutually independent. Once we move to a stationary time series environment the independence assumption is undesirable. In this more general setting, the finiteness of $E[T(\bar{v}_T - \mu)^2]$ requires assumptions about the autocovariances of v_t as the next theorem shows.

Theorem 2 WLLN for covariance stationary processes
Let $\{v_t,\ t = -\infty, \ldots, -1, 0, 1, \ldots \infty\}$ *be a covariance stationary process with mean* $E[v_t] = \mu$ *and autocovariances* $E[(v_t - \mu)(v_{t-i} - \mu)] = \gamma_i$ *for* $i = 0, 1, \ldots$. *If* $\sum_{i=0}^{\infty} |\gamma_i| < \infty$ *then* $T^{-1}\sum_{t=1}^{T} v_t \xrightarrow{p} \mu$.

Proof: As with Theorem 1, the proof rests on an application of Chebyshev's inequality. As before, we use this inequality to establish that

$$\lim_{T\to\infty} P[|\bar{v}_T - \mu| \ \geq \ \epsilon] \ \leq \ \lim_{T\to\infty} \frac{E[(\bar{v}_T - \mu)^2]}{\epsilon^2} \tag{14}$$

and that the right hand side of (14) is zero for all $\epsilon > 0$. To establish the latter in this more general setting, we must examine $E[(\bar{v}_T - \mu)^2]$ under the conditions of Theorem 2. As before, we can deduce that

$$E[(\bar{v}_T - \mu)^2] \ = \ T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T} E[(v_t - \mu)(v_s - \mu)] \tag{15}$$

From the covariance stationarity of v_t, it follows that

$$T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T} E[(v_t - \mu)(v_s - \mu)] \ = \ T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma_{|t-s|} \tag{16}$$

Collecting terms, it can be shown that

$$T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma_{|t-s|} \ = \ T^{-1}\left\{\gamma_0 \ + \ 2\sum_{i=1}^{T-1}\left(\frac{T-i}{T}\right)\gamma_i\right\} \tag{17}$$

Noting that $0 < (T-i)/T < 1$ for $i = 1, 2, \ldots T-1$, it can be seen that

$$T^{-1}\left\{\gamma_0 \ + \ 2\sum_{i=1}^{T-1}\left(\frac{T-i}{T}\right)\gamma_i\right\} \ \leq \ |T^{-1}\left\{\gamma_0 \ + \ 2\sum_{i=1}^{T-1}\left(\frac{T-i}{T}\right)\gamma_i\right\}| \tag{18}$$

$$\leq \; T^{-1}\left\{|\gamma_0| + 2\sum_{i=1}^{T-1}\left(\frac{T-i}{T}\right)|\gamma_i|\right\} \tag{19}$$

$$\leq \; T^{-1}2\sum_{i=0}^{T-1}|\gamma_i| \tag{20}$$

Combining (15)-(20), it follows that

$$E[(\bar{v}_T - \mu)^2] \;\leq\; T^{-1}2\sum_{i=0}^{T-1}|\gamma_i| \tag{21}$$

Combining (14) and (21), it follows that

$$\lim_{T\to\infty} P[|\bar{v}_T - \mu| \;\geq\; \epsilon] \;\leq\; \lim_{T\to\infty}\left\{\frac{2\sum_{i=0}^{T-1}|\gamma_i|}{T\epsilon^2}\right\} \;\leq\; \lim_{T\to\infty}\left\{\frac{2\sum_{i=0}^{\infty}|\gamma_i|}{T\epsilon^2}\right\} \tag{22}$$

The desired result then follows because of the assumption that $\sum_{i=0}^{\infty}|\gamma_i| < \infty$. ◇

If the autocovariances satisfy the condition $\sum_{i=0}^{\infty}|\gamma_i| < \infty$ then they are said to *absolutely summable*. By the nature of autocovariances, it can be seen that if they are absolutely summable then this is placing a condition on the memory of the series. Specifically, the autocovariances must decay to zero.