Solutions to Practice Problem Set 5

1. Using the Triangle inequality we have

$$\begin{aligned}\|S\| &= \|\Gamma_0 + \sum_{i=1}^{\infty}(\Gamma_i + \Gamma_i^{'})\| \\ &= \|\Gamma_0\| + \sum_{i=1}^{\infty}(\Gamma_i + \Gamma_i^{'})\| \\ &= \|\Gamma_0\| + \sum_{i=1}^{\infty}(\| \Gamma_i\| + \|\Gamma_i^{'}\|) \\ &= \sum_{i=0}^{\infty}\|\Gamma_i\| + \sum_{i=1}^{\infty}\|\Gamma_i^{'}\| \qquad (1)\end{aligned}$$

Now $tr(AB) = tr(BA)$ and so $\| \Gamma_i\| = \|\Gamma_i^{'}\|$. Using this property and $\|.\| \geq 0$, (1) yields

$$\begin{aligned}\|S\| &= \sum_{i=0}^{\infty}\|\Gamma_i\| + \sum_{i=1}^{\infty}\|\Gamma_i\| \\ &\leq 2\sum_{i=0}^{\infty}\|\Gamma_i\|\end{aligned}$$

and so $\sum_{i=0}^{\infty}\|\Gamma_i\| < \infty$ implies $\|S\| < \infty$.

2. If $q = 1$ then $\|\Gamma_i\| = |\gamma_i|$ (see Practice Problem set 4 Question 3) where γ_i is the i^{th} autocovariance of f_t. The condition, therefore becomes $\sum_{i=0}^{\infty}|\gamma_i| < \infty$ which is the requirement that the autocovariances be absolutely summable.

3(a) If $q = 1$ then from equation (1) on the question sheet

$$\begin{aligned}T\hat{S}_{TR} &= \sum_{t=1}^{T}\hat{f}_t^2 + \sum_{i=1}^{\ell}\{\sum_{t=i+1}^{T}\hat{f}_t\hat{f}_{t-i} + \sum_{t=i+1}^{T}\hat{f}_{t-i}\hat{f}_t\} \\ &= \sum_{t=1}^{T}\hat{f}_t^2 + 2\sum_{i=1}^{\ell}\sum_{t=i+1}^{T}\hat{f}_t\hat{f}_{t-i}\end{aligned}$$

Let $\hat{f}$ be the $T \times 1$ vector with t^{th} element $\hat{f}_t$ and D_j be the $T \times T$ matrix with $m - n^{th}$ element $D_j(m,n) = \delta_j(m,n)$ where

$$\begin{aligned}\delta_j(m,n) &= 1, \text{ for } m = n + j,\ n = 1, 2, \ldots T - j, \text{ or } n = m + j,\ m = 1, 2 \ldots T - j \\ &= 0, \text{ else}\end{aligned}$$

Using these definitions, it follows that

$$\sum_{t=1}^{T} \hat{f}_t^2 = \hat{f}' I_T \hat{f}$$
$$2 \sum_{t=i+1}^{T} \hat{f}_t \hat{f}_{t-i} = \hat{f}' D_i \hat{f}$$

Therefore, it follows from (1) that

$$T\hat{S}_{TR} = \hat{f}' I_T \hat{f} + \sum_{i=1}^{\ell} \hat{f}' D_i \hat{f} = \hat{f}' (I_T + \sum_{i=1}^{\ell} D_i) \hat{f}$$

the proof is completed by noting that $D = I_T + \sum_{i=1}^{\ell} D_i$.

3(b) A suitable program for this and other parts of this question can be downloaded from the class web page. The eigenvalues of D are: -0.0000, 0.0000,-1.0000, 0.5929, 0.7892, -0.6554, -1.1444, 1.4490, 3.8662, 6.1024. Therefore, D is indefinite.

3(c) In my simulations k is negative in 10.1 percent of the replications. The exact number will depend on the sequence of random numbers.

3(d) Recall that the Bartlett kernel is:

$$\begin{aligned} \omega_{i,T} &= 1 - \frac{i}{b_T + 1}, \text{ for } i \leq b_T \\ &= 0 \text{ else.} \end{aligned}$$

So if $b_T = 3$ and $q = 1$ we obtain:

$$\hat{S}_{HAC} = T^{-1} \sum_{t=1}^{T} \hat{f}_t^2 + 2\left(\frac{3}{4}\right) T^{-1} \sum_{t=2}^{T} \hat{f}_t \hat{f}_{t-1} + 2\left(\frac{2}{4}\right) T^{-1} \sum_{t=3}^{T} \hat{f}_t \hat{f}_{t-2} + 2\left(\frac{1}{4}\right) T^{-1} \sum_{t=4}^{T} \hat{f}_t \hat{f}_{t-3}$$

Using similar arguments to part (b) it can be shown that $\hat{S}_{HAC} = T^{-1} H'CH$. The eigenvalues of C are: 0.1123, 0.0971, 0.0532, 0.2017, 0.2500, 0.3088, 0.8133, 1.7153, 2.7864 3.6618. Therefore C is positive semi-definite.

3(e) k is negative zero percent of the replications.