Acoustic Wave Propagation

Adam Attarian

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1 Motivation and Derivations

The objective of this project is to understand the different boundary conditions and the resulting effect on low-frequency acoustical wave propagation in a length of PVC pipe. The two boundary conditions tested were a foam end-cap as well as a metal plate. Data were also collected for an open ended/no boundary condition case to aid in computation. The data collected were then used to obtain reflection and amplitude coefficients over a range of frequencies. The reflection coefficients in turn, are used to estimate the unknown physical parameters in the mathematical model for the boundary conditions.

1.1 Wave Equation: Overview

The motion and propagation of relatively low frequency acoustic waves through a fluid is given by the classical one-dimension linear wave equation:

\[ \frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}, \quad x \in (0, l) \]  

(1.1)

Where \( \phi \) is the velocity potential, \( c \) is the speed of sound through the medium and \( l \) is the length of the pipe. The motion of the wave is presumed to be one dimensional. The wave equation has many solutions given the specific boundary conditions, however one particularly useful (and elegant) solution is the D’Alembert Solution, given by:

\[ \phi(t, x) = F \left( t - \frac{x}{c} \right) + G \left( t + \frac{x}{c} \right) \]

(1.2)

Where \( F \) and \( G \) are arbitrary functions corresponding to a forward traveling wave and a reflected, backwards traveling wave, respectively. The acoustical wave motion in a fluid can also be described by the acoustical pressure, \( p \). The acoustical pressure is related to the velocity potential by \( p(t, x) = \rho \cdot \phi_t \), and satisfies the wave equation given by (1.1). For this project, measurements of the acoustical pressure \( p \) were taken.
1.2 Oscillating Boundaries

The first type of boundary condition we will consider is oscillating boundaries. If we assume a damped harmonic oscillator, then the interaction between the boundary $x = l$ and the internal pressure can be modeled as

$$m\delta_{tt} + d\delta_t + k\delta = -\rho \frac{\partial}{\partial t} \phi(t, l)$$  \hspace{1cm} (1.3)

Where $\delta$ is the normal displacement of the boundary condition in the direction of the interior fluid. Note that the left side of the equation is the standard spring-mass dashpot system. The boundary plate can be thought to vibrate in accordance with that model. We will also assume that the fluid does not penetrate the boundary surface. That is,

$$\delta_t(t) = \phi_x(t, l)$$  \hspace{1cm} (1.4)

Using the interior pressure model (1.3) and the preceding condition along with the wave equation solution (1.2) we now have

$$\frac{\partial \phi}{\partial x}(t, l) = -\frac{1}{c} \frac{\partial F}{\partial x}(t - \frac{l}{c}) + \frac{1}{c} \frac{\partial G}{\partial x}(t + \frac{l}{c})$$

Integrating we obtain

$$\delta(t) = -\frac{1}{c} [F(t - l/c) - G(t + l/c)]$$  \hspace{1cm} (1.5)

With out loss of generality we may assume that the constant of integration is zero. By setting $\tilde{F} = F(t - l/c)$ and $\tilde{G} = G(t + l/c)$, the above equation now becomes

$$\delta(t) = -\frac{1}{c} [\tilde{F}(t) - \tilde{G}(t)]$$  \hspace{1cm} (1.6)

Now substituting (1.6) into (1.3) produces

$$m\tilde{G}'' + (d + \rho c)\tilde{G}' + k\tilde{G} = m\tilde{F}'' + (d - \rho c)\tilde{F}' + k\tilde{F}$$  \hspace{1cm} (1.7)

We now assume that the incident wave $\tilde{F}$ to the boundary $x = l$ generated by a harmonic (e.g sinusoidal) input at $x = 0$ is a simple harmonic of $\omega/2\pi$ with $\omega$ being the angular frequency in radians per second. That is,

$$\tilde{F}(t) = A_0 \exp(j\omega t)$$  \hspace{1cm} (1.8)

And so the right hand side of (1.7) is a harmonic forcing function. As a result, the steady state solution of (1.7) is also harmonic with the same frequency.
\[ \tilde{G}(t) = R(\omega)A_0 \exp(j\omega t) \]  

(1.9)

Where the complex coefficient, \( R(\omega) \) is the reflection coefficient. Consider (1.8) and (1.9) and their respective derivatives:

\[ \tilde{G}' = j\omega R(\omega)A_0 \exp(j\omega t) \]  

(1.10)

\[ \tilde{G}'' = -\omega^2 R(\omega)A_0 \exp(j\omega t) \]  

(1.11)

\[ \tilde{F}'(t) = j\omega A_0 \exp(j\omega t) \]  

(1.12)

\[ \tilde{F}''(t) = -\omega^2 A_0 \exp(j\omega t) \]  

(1.13)

Using these derivatives we can substitute into (1.7) and obtain a relation for the reflection coefficient for the oscillating boundary condition to obtain

\[ R(\omega) = \frac{m\omega^2 - j(d - \rho c)\omega - k}{m\omega^2 - j(d + \rho c)\omega - k} \]  

(1.14)

Using collected data, we will be able to formulate an inverse least squares problem to estimate the parameters \( m, d, \rho, \) and \( k. \)

1.3 Damped Elastic Boundaries

The second type of boundary condition that will be considered is damped elastic boundaries. These can be thought of as pliable and porous, similar to foam. With respect to the acoustical pressure \( p \), the boundary condition has the form of

\[ \alpha p(t, l) + \beta p_t(t, l) + cp_x(t, l) = 0 \]  

(1.15)

Using \( p(t, x) = \rho \phi_t \) as well as derivatives calculated for \( \tilde{F} \) and \( \tilde{G} \) we can develop the reflection coefficient. Equation (1.15) now becomes

\[ \alpha \rho (\tilde{F}' + \tilde{G}') + \beta \rho (\tilde{F}'' + \tilde{G}'') + c \rho (\frac{1}{c} \tilde{F}'' + \frac{1}{c} \tilde{G}'') = 0 \]

which simplifies to

\[ \alpha j + \alpha jR(\omega) - \beta \omega - \beta \omega R(\omega) + \omega - \omega R(\omega) = 0 \]

Assuming harmonic incident wave as previously, we obtain the damped elastic reflection coefficient

\[ R(\omega) = \frac{j\omega(1 - \beta) - \alpha}{j\omega(1 + \beta) + \alpha} \]  

(1.16)

3
1.4 Project Description

This project involves the following steps:

1. The acoustic pressure at any given location in the pipe for planar wave propagation is given by:

\[ p(t, x) = A(\omega) \exp(j\omega(t - x/c)) + A(\omega)R(\omega) \exp(j\omega(t + x/c)) \]  

By measuring the pressure \( p(t, x_k) \) using microphones placed at several axial locations \( x_k \) and for a specific angular frequency \( \tilde{\omega} \), an inverse least squares problem can be formulated to estimate both complex coefficients, \( A(\tilde{\omega}) \) and \( R(\tilde{\omega}) \) using the following cost functional

\[ J_\phi = \sum_{k=1}^{2001} \sum_{n=1}^{3} (|\phi_d(t_k, x_n; \omega_i)| - |\phi_m(t_k, x_n; \omega_i)|)^2 \]  

Where \( \phi_d \) represents the collected data and \( \phi_m \) is the data set constructed by (1.17) at the \( k^{th} \) time sample and the \( n^{th} \) axial location for a given frequency \( \omega_i \). The DSP operated at a sampling frequency \( f_s = 2kHz \) over a one second interval providing 2000 data points. Considering both a metallic plate and foam boundary at \( x = l \), as well as an open ended case, the sets of data can be used to estimate \( R(\tilde{\omega}) \). The data collected are denoted \( \phi_d(\tilde{\omega}) \) over the range \( f \in [100, 200] \) where \( f \) is in Hertz.

2. In order to determine the physical parameters for each boundary condition described by (1.14) and (1.16), an inverse least square problem is formulated with the cost functional

\[ J_R = \sum_{i=1}^{11} (|R_d(\omega_i)| - |R_m(\omega_i)|)^2 \]  

Where \( R_d \) represents the \( R(\omega) \) coefficients determined by (1.18) and \( R_m \) the model \( R(\omega) \) coefficients defined by either (1.14) and (1.16). This minimization will allow us to determine the set of physical parameters \( (m, d, k, \rho) \) and \( (\alpha, \beta) \) that best fits the physical data.

2 Data and Results

2.1 Overview

As stated before, to collect data we used a simple harmonic oscillator attached to one end of a PVC pipe with microphones located across the axial distance. To get a better idea of the kind of data that we are working with, it is useful to see a time domain representation
of the wave so that we can see how noisy or how much variance there is in the data. The first figure is the $f = 160$Hz wave for all three boundary condition cases. The frequency for display is arbitrary, and is used just to get a sense of the data. The quality of the data will be discussed in a later section.

![Figure 1: Time Domain Representation of Data](image)

### 2.2 Coefficient Determination

The first computation was to calculate $A(\omega_i)$, the Amplitude Coefficient for a given frequency. As seen in (1.17), $A(\omega)$ is not dependant on the reflected waveform and so the open ended case was used in the calculation. The computation was relatively straight forward using `fminsearch` to minimize a modified cost functional of (1.18) in MATLAB. Using the substitution $f = \frac{\omega}{2\pi}$ we obtained the following results. The magnitude of the coefficient is presented since our collected data is only real-valued.
A plot of $|A(f)|$ is presented below.

![Amplitude Coefficient Magnitude for All Cases](image)

**Figure 2: Amplitude Coefficients**

The red data points indicate the value at the given frequency, and the blue trace is an interpolation between the sampled frequencies. This applies to all further plots, as well. Now that we have $A(\omega)$ well defined for all frequencies, we can reference these values for the computation of $R(\omega)$ for both the hard wall and foam boundary conditions. To determine the Reflection Coefficient for both cases, (1.18) was again minimized using `fminsearch` in MATLAB yielding the following data and plot for the hard wall metallic plate boundary.
| \( f \) (Hz) | \(| R(f) |\) (hard wall) |
|-----------|----------------------|
| 100       | 3.2258               |
| 110       | 3.667                |
| 120       | 1.8463               |
| 130       | 0.4489               |
| 140       | 0.2634               |
| 150       | 0.2044               |
| 160       | 0.0936               |
| 170       | 0.8505               |
| 180       | 2.0299               |
| 190       | 4.0927               |
| 200       | 3.6368               |

Figure 3: Reflection Coefficient for Hard Wall Boundary
Last was the calculation of the \( R(\omega) \) coefficients for the foam barrier, which were computed in an identical manner to the hard-wall boundary.

\[
\begin{array}{|c|c|}
\hline
f \text{ (Hz)} & |R(f)| \text{ (foam)} \\
\hline
100 & 0.5621 \\
110 & 1.0529 \\
120 & 1.3565 \\
130 & 0.8791 \\
140 & 0.4036 \\
150 & 0.4562 \\
160 & 0.5894 \\
170 & 0.7098 \\
180 & 0.8519 \\
190 & 1.2156 \\
200 & 1.4120 \\
\hline
\end{array}
\]

Figure 4: Reflection Coefficient for Foam

Speaking computationally, the minimization of (1.18) have produced a set of frequency-stacked complex \( A(\omega) \) and \( R(\omega) \) vectors which can now be compared to our models for both oscillating and damped elastic boundaries. In doing so we can isolate the set of physical parameters to reproduce the data set uniquely.
2.3 Parameter Determination

Recall the model for the oscillating boundary, (1.14).

\[
R(\omega) = \frac{m \omega^2 - j(d - \rho c) \omega - k}{m \omega^2 - j(d + \rho c) \omega - k}
\]

And the corresponding damped elastic model, (1.16).

\[
R(\omega) = \frac{j \omega (1 - \beta) - \alpha}{j \omega (1 + \beta) + \alpha}
\]

Because we now have frequency stacked \( R(\omega) \) values from collected data, we can use \texttt{fminsearch} to minimize (1.19) for an ordered set of frequencies, namely 100 to 200 Hz in 10 Hz increments. Of interest is if in fact the damped elastic model is a good approximation for the foam based data or if the oscillating boundary model fits the foam data better. Conversely, is it possible that damped elastic model fits the plate data better than the oscillating boundary model does? Evaluating the cost functional at the approximated values will reveal which model provides the best fit for the given data set.

The first minimization was with the data collected from the metal plate boundary and the oscillating boundary model. Performing the computation in the neighborhood of 0.1 for all values yielded:

\[
\begin{bmatrix}
m \\
d \\
k \\
\rho \\
\end{bmatrix} = \begin{bmatrix}
\approx 0 \\
0.1435 \\
0.1503 \\
-0.0014 \\
\end{bmatrix}, \quad J_R = 24.72
\]

(2.1)

The next minimization was the foam barrier data with the oscillating boundary model to see how well it fits data it necessarily shouldn’t. Again, the minimization was calculated in the neighborhood of 0.1 for all parameters.

\[
\begin{bmatrix}
m \\
d \\
k \\
\rho \\
\end{bmatrix} = \begin{bmatrix}
-0.0148 \\
2.2565 \\
1.0433 \\
0.0291 \\
\end{bmatrix}, \quad J_R = 1.230
\]

(2.2)

Shifting to minimizing the damped elastic boundary model, we obtain the following parameters for the data collected with the foam boundary when minimizing in the neighborhood of 0.1.

\[
\begin{bmatrix}
\alpha \\
\beta \\
\end{bmatrix} = \begin{bmatrix}
0.001 \\
0.0737 \\
\end{bmatrix}, \quad J_R = 1.2711
\]

(2.3)

Applying the damped elastic boundary model to the data collected with the metal plate, the parameters are found to be

\[
\begin{bmatrix}
\alpha \\
\beta \\
\end{bmatrix} = \begin{bmatrix}
0 \\
-0.2984 \\
\end{bmatrix}, \quad J_R = 24.72
\]

(2.4)
To obtain a better idea of how well the model’s results fits the experimental data, we can generate a new data set using the calculated parameters and plot this along side the experimental data set. In all plots, the dark red trace represents the output of the given model with the calculated physical parameters.

Figure 5: Model Generated by both (2.1) and (2.4), Plate Barrier

Figure 6: Model Generated by (2.2), Foam Barrier
3 Discussion & Conclusion

3.1 Analysis

Qualitatively, the model doesn’t appear to fit the data very well. The model for damped elastic boundary should produce a line of constant slope for the magnitude of $R(\omega)$ however the collected data reveals only a slight linear trend for a portion of the intervals. The data for hard wall boundary has a parabolic shape, whereas this shape is not predicted by the model.

The cost functional (1.19) did minimize the sum of square error for all cases, and in the hard wall boundary condition resulted in $|R(\omega)|$ equal to the average value of the magnitude of the experimental data points. Only in the case of the foam boundary data did the different boundary condition models make a quantitative difference, and the difference was very small at that. Interestingly, the oscillating boundary condition model modeled the foam case better, albeit slightly, than the damped elastic model. In both cases, the model approximated the average value for the sample data.

There could be several reasons why the models aren’t fitting the shape of our experimental data. Among them:

1. Quality of Data: Additional frequency components, DC shift, AWGN and Environmental Noise, frequency response and calibration of sensors

2. Minimization Techniques: Different cost functionals will minimize different kinds of noise, Non-uniqueness of minimizing parameters.

3. Robustness of Model: How well the model can handle the inherent AWGN Noise in the system.
Each of these points will be addressed further.

3.2 Quality of Data

Looking at the time domain representation of the data, a few things are evident. First, there is a sizable DC shift in all of the data for all cases and frequencies. Generally this is not considered ideal and can skew results. We attempted to compensate for this in the code shifting all data by the mean, which should center the signals about zero however some residual DC value may remain. This is an important fact, because our models will generate zero-mean sinusoids, and so problems can arise when trying to minimize the difference between the zero-mean model and experimental data with a DC constant. Also by looking at the data you notice how out of phase the data is with a standard sinusoid. This too may cause difficulties in minimization.

Another possibility was the presence of additional frequency components in the data. If additional frequencies were present, there would be an inescapable difference between the model, which generates only one specific frequency, and the data. To check for other frequency components, a 4096-point Fast Fourier Transform was run on an arbitrarily chosen frequency ($f = 150$Hz), location ($x = 0.945$m), and boundary condition (foam). The resulting frequency domain representation is shown below.

![Frequency Domain Representation](image)

Figure 8: Fast Fourier Transform of $f = 150$Hz

Clearly, near all of the power in the signal is isolated directly at the center frequency and DC, thus ruling out the presence of any other signal that may skew the results. The lack of a peak at 60Hz makes sense, as the 60Hz background hum of the fluorescent lights and AC power wouldn’t have made a difference on the inside of the pipe; it would have most likely been blocked or absorbed by the PVC material. Additionally, we did our best to keep quiet during data collection. The result from the FFT also shows that the processor was outputting the correct frequency, eliminating another point of variation from the model.

The frequency response of the microphones are another degree of variation in the system. Even if all of the microphones are well conditioned to the generated frequencies, the variation between response across the different microphones axially could contribute to noise in the data that the model could not account for. This is most likely not a problem,
as most commercially available microphones have an acceptable frequency response for
the frequency range of this experiment and we presume quality control is good enough to
ensure consistent results across all sensors.

Generally, the foam case fits the model with a reasonable correlation for more than half
of the tested frequencies ($f \geq 140$Hz). The same can not be said for the metal case. This
could be due to the frequency response of the metal in relation to the acoustic wave. It is
possible that a range of frequencies (for example, $f \in [130, 160]$) within the test produces
behavior from the metal plate that is substantially different from other frequencies. The
spring-mass dashpot model does take into account the physical properties of the material,
but whether or not this translates into adjusting for frequency response is unclear.

3.3 Minimization Techniques

In all inverse least squares problems, a cost functional is defined minimized to isolate
model parameters that when used, best reproduce the collected data. Because our model
produces complex-valued results and our data is completely real valued, how we define the
cost functionals is a non-trivial exercise. Recall the two cost functionals (1.18) and (1.19):

$$J_\phi = \sum_{k=1}^{2001} \sum_{n=1}^{3} (|\phi_d(t_k, x_n; \omega_i)| - |\phi_m(t_k, x_n; \omega_i)|)^2$$

$$J_R = \sum_{i=1}^{11} (|R_d(\omega_i)| - |R_m(\omega_i)|)^2$$

These functionals are minimizing the square of the difference between the data and model,
referred to as the 2-norm. In Control and Optimization, there can be many different types
of cost functionals dependent on the type of error that is to be minimized. For example, in
chemical engineering it is often desirable to minimize long term error, so a cost functional is
chosen suited to that purpose, but may not minimize short term errors and perturbations.

In testing of the code, we found that there were several sets of parameters that could
produce the same set of data at the same cost. The set of parameters depended on the
neighborhood in which fminsearch was run. While this shouldn’t necessarily effect results
as the function was minimal at the result, it does add another degree of freedom to the
computational process.

During the development of our code, we tried numerous variants of (1.18) and (1.19),
including taking the absolute value after the taking difference, or moving the square to the
individual terms. These modifications produced slightly different results, but the general
shape of the data remained the same. In calculating the physical parameters we tried using
the 1-norm and the infinity-norm of (1.19) but the results were identical. This makes sense,
as there is no way a linear model will be able to well-approximate a parabolically-shaped
data set. The only way the system could minimize error was to return a result that was
equal to the mean value of the data set.
3.4 Model Robustness

The largest factor affecting a sound-based model and the system can be the presence of additive white gaussian noise. Noise in the system can come from many different places. This noise is nearly inescapable in the laboratory and signal processor, and the only thing that can be done is to develop a model that can handle any given noise well, or at least determine how well a given model can handle noise so that you will be able to better interpret the results. This can be done by using the model to generate a set of data, add varying levels of AWGN, and then run the inverse program to check out well the output matches the known input. Unfortunately for this model and run of code we did not verify the elasticity of the model with respect to AWGN, so we do not know how well the model handles noise and it’s effect on parameter and coefficient determination.

Whenever a model doesn’t fit the data, and even when it does, noise must be considered. However in this case it is unlikely that noise is the sole reason for the ill-fit. Typically the variance in the data induced by AWGN will be small relative to the data. Low variance AWGN could not explain the relative high variance of the metal plate boundary data. If the variance were high then the data could be explained solely from noise, but it is unlikely that this is the case. The system environment is well kept, external vibrations were kept to a minimum, and the signal processing equipment was of high quality. Further, examining the FFT of the data shows a nearly zero baseline noise level at frequencies outside the generated frequency suggesting further that noise is less of an issue than one would think in a sound-based experiment. While low levels of noise may be ignored by the FFT, high levels would be shown as random peaks at random and all frequencies.

3.5 Conclusion

After running the simulations, we found that models for both the oscillating and damped elastic boundary condition provide nearly identical results when run against data from both metal plate and foam boundaries. Using the oscillating boundary condition model with the foam data generated a Sum Of Squares Error slightly less than the competing model, but both of the models provided results nearly, if not identical, to the mean value of the data.

In general, the foam boundary condition produced reflection coefficients of lesser magnitude than the metal plate boundary, implying that the foam damps the amplitude of the reflected wave more so than the metal plate does. Additionally, the models do not seem to fit the data in general. The data for the metal plate boundary is parabolic in shape whereas the model provides only a linear fit in the best cases. The foam boundary data is slightly more well behaved and is linear in shape for over half of the sampled frequencies, however the model again approximates it using the mean value of the data.

It is unlikely that environmental factors, intrinsic physical properties, or large amounts of additive white gaussian noise are to blame for the ill-fit. In short, to have a better model
we need more data. Data were only collected over a 100Hz interval. I would be curious to see the trend in upwards of 500-800Hz – then we would able to see if the ill-matched data points are locally anomalous to the frequency and are not actually indicative of the general fit of the model to larger amounts of data.