MA 426 Test 2 "Review"

1. Definitions

   (a) Continuity of a function f at x_0: We say $f : A \subseteq M \to N$ is **continuous** at $x_0 \in A$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\|x - x_0\|_M < \delta \Rightarrow \|f(x) - f(x_0)\|_N < \epsilon$

   (b) Relatively Open: A subset B of A is **relatively open** iff $\exists$ an open set $U \subseteq M$ such that $B = A \cap U$.

   (c) Relatively Closed: $A \subseteq M$. A subset B of A is **relatively closed** iff $\exists$ a closed subset $F \subseteq M$ such that $B = A \cap F$.

   (d) A subset $A \subseteq M$ is **path connected** iff $\forall p, q \in A$ there exists a path γ in M from p to q such that $\gamma([a,b]) \subseteq A$. The path $\gamma : [a,b] \to M$ is a **path** from $p = \gamma(a)$ to $q = \gamma(b)$.

   (e) The Derivative: Assume E, F are normed linear spaces and $U \subseteq E$ an open set. $f : U \to F$. We say f is **differentiable** at $x_0 \in U$ iff $\exists$ a continuous linear map $L_{x_0} : E \to F$ such that
$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - L_{x_0}(h)\|_F}{\|h\|_E} = 0.$$

2. Proofs to know!

   (a) Let M, N be metric spaces, $A \subseteq M$ and $f : A \to N$. Then the following are equivalent.

      i. f is continuous on A.
      ii. for each sequence $\{x_n\} \in A$ such that $x_k \to x_0 \in A$ it follows that $f(x_k) \to f(x_0)$.
      iii. for each open subset $U \subseteq N$, $f^{-1}(U)$ is relatively open in A.
      iv. for each closed subset $F \subseteq N$, $f^{-1}(F)$ is relatively closed in A.

   Proof. (i) → (ii): Assume $\{x_k\}$ is a sequence in A such that $x_k \to x_0 \in A$. Let $\epsilon > 0$. Since f is continuous at x_0 $\exists \delta > 0$ such that $d_M(x, x_0) < \delta, x \in A \Rightarrow d_N(f(x), f(x_0)) < \epsilon$. $\boxed{\text{Choose } N_0 \in \mathbb{N}}$ such that $k \geq N_0 \Rightarrow d_M(x_k, x_0) < \delta$. Notice that $\boxed{k \geq N_0}$
   $\Rightarrow d_M(x_k, x_0) < \delta$
   $\therefore d_N(f(x_k), f(x_0)) < \epsilon \Rightarrow f(x_k) \to f(x_0)$.

   (ii) → (iv): Let $F \subseteq N$ be closed. Show $f^{-1}(F)$ is relatively closed. Know that $f^{-1}(F) \subseteq A \subseteq M$. Let $\{b_k\}_1^\infty$ be any sequence in $B = f^{-1}(F)$ such that $b_k \to a_0 \in A$. Must prove that $a_0 \in B = f^{-1}(F)$ (satisfies a lemma). Since $b_k \to a_0$ is follows that $f(b_k) \to f(a_0)$, but $f(b_k) \in F$ (since $b_k \in f^{-1}(F)$). Since $\{f(b_k)\}_1^\infty$ is a sequence in the closed set F and $f(b_k) \to f(a_0)$ it follows that $f(a_0) \in F$.
   $\therefore a_0 \in f^{-1}(F)$, So $f^{-1}(F)$ is relatively closed.

(iv) → (iii): Let $U \subseteq N$ be open. Show that $f^{-1}(U)$ is relatively open in A. Let $F = N \backslash U$. F is closed in N. Therefore $f^{-1}(F)$ is relatively closed in $A \Rightarrow f^{-1}(U) = f^{-1}(N \backslash F) = A \backslash f^{-1}(F) = A \backslash f^{-1}(C \cap A) = (A \backslash C) \cup (A \backslash A) = A \backslash C = A \cap (M \backslash C) = A \cap (\text{some open set})$.

(iii) → (i): Show continuous at any $x_0 \in A$. Let $\epsilon > 0$ then $D_\epsilon(f(x_0))$ is open in N. Therefore $f^{-1}(D_\epsilon(f(x_0)))$ is relatively open in A, so there exists an open set $U \subseteq M$ such that $f^{-1}(D_\epsilon(f(x_0))) = U \cap A$. Since $x_0 \in f^{-1}(D_\epsilon(f(x_0)))$ we have $x_0 \in U \cap A$. Since $x_0 \in U$ and U is open in M there exists $\delta > 0$ such that $D_\delta(x_0) \subseteq U$. $\Rightarrow$ If $x \in A$ such that $d_M(x, x_0) < \delta$ then $x \in D_\delta(x) \cap A \subseteq U \cap A = f^{-1}(D_\epsilon(f(x_0)))$ and so $f(x) \in D_\epsilon(f(x_0))$.
$\therefore d_N(f(x), f(x_0)) < \epsilon$ ☐

(b) Let $f : A \to Y$ be continuous. If $B \subseteq A$ is compact then $f(B)$ is compact.

Proof. Let $\{U_i\}$ be an open cover of $f(B)$. Since f is continuous $f^{-1}(U_i)$ is relatively open, and $f^{-1}(U_i) = A \cap U_i'$, where $U_i' \subseteq M$. Need to show that U_i' covers B.

> Let $y \in B$ then $f(y) \in f(B)$, so $f(y) \in \bigcup_{i \in I} U_i \Rightarrow f(y) \in U_j$ for some $J \in I$. Therefore $y \in f^{-1}(U_j) = U_j' \cup A \Rightarrow y \in U_j'$ Since y is arbitrary and B compact, you have $B \subseteq \bigcup_{k=1}^{N} U_{i_k}'$.

Now let $z \in f(B)$. Then $z = f(y)$ for some $y \in B \subseteq A$. Since $y \in B$ and B compact, $y \in U_{i_l}'$ for some $1 \leq l \leq N$. Then $y \in A \cap U_{i_l}' = f^{-1}(U_{i_l})$. So now $z = f(y) \in U_{i_l}$. Since $z \in f(B)$ is arbitrary and contained in a single subcover, $f(B) \subseteq \bigcup_{k=1}^{N} U_{i_k}$
$\therefore f(B)$ is compact. ☐

(c) The composition of two continuous functions is continuous. Or, more technically: Let M, N, P metric spaces and $f : A \subseteq M \to N$ and $g : B \subseteq N \to P$ and $f(A) \subseteq B$. Show $(g \circ f)(x) : A \subseteq M \to P$ is continuous.

Proof. Let the sequence $\{x_k\} \in A$ converge to $x_0 \in A$. Since f is continuous, $f(x_k) \to f(x_0)$. As $f(A) \subseteq B \Rightarrow \{f(x_k)\}, f(x_0) \in B$. Since g continuous, $\{g(f(x_k))\} \to g(f(x_0)) \in P \Rightarrow (g \circ f)(x_k) \to (g \circ f)(x_0)$
$\therefore g \circ f : A \to P$ is continuous on A. ☐

(d) Let M be a metric space and $f : M \to \mathbb{R}$ a continuous function. If $K \subseteq M$ is compact then $\exists\ x_*, x^* \in K$ such that $f(x_*) \leq f(x) \leq f(x^*)$.

Proof. Notice that K is compact, so $f(K)$ is also compact; therefore $f(K)$ is bounded. Let $M = \sup(f(K))$. For each $n \in \mathbb{N}$, $M - \frac{1}{n} < M$ is not an upper bound of $f(K)$.

So $\exists\ y_n \in f(K)$ such that $(M - \frac{1}{n}) < y_n \leq M \Rightarrow 0 \leq |y_n - M| \leq \frac{1}{n}$. By squeezing, $y_n \to M$. Since $f(K)$ is closed, $M \in \mathrm{cl}(f(K)) = f(K)$.
$\therefore M = f(x^*)$ for some $x^* \in K$. Use a similar argument for $\inf(f(K))$ to reach the big conclusion. ☐

(e) If A is an open subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ is differentiable at $x_0 \in A$ then the partials of f exist at x_0 and $Df(x_0)(h) = \nabla f(x_0) \cdot h$

Proof. Assume f is differentiable at x_0. Then

$$\begin{aligned}
\lim_{te_i \to 0} \frac{\|f(x_0 + te_i) - f(x_0) - Df(x_0)(te_i)\|}{\|te_i\|} &= 0 \\
\lim_{t \to 0} \left| \frac{f(x_0 + te_i) - f(x_0) - t \cdot Df(x_0)(e_i)}{t} \right| &= 0 \\
\lim_{t \to 0} \left| \frac{f(x_0 + te_i) - f(x_0)}{t} - Df(x_0)(e_i) \right| &= 0 \\
\lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t} &= Df(x_0)(e_i)
\end{aligned}$$

Notice that

$$\begin{aligned}
Df(x_0)(h) &= Df(x_0)\Big(\sum h_i e_i\Big) \\
&= \sum Df(x_0)(h_i e_i) \\
&= \sum h_i \cdot Df(x_0)(e_i) \\
&= \sum h_i \cdot \frac{\partial f}{\partial x_i}\Big|_{x_0} \\
&= \nabla f(x_0) \cdot h
\end{aligned}$$

$\therefore \frac{\partial f}{\partial x_i}\big|_{x_0}$ exists and $\frac{\partial f}{\partial x_i}\big|_{x_0} = Df(x_0)(e_i)$. ☐