Final Exam Prep (New)

1. Definitions

   (a) The Derivative: Assume E, F are normed linear spaces and $U \subseteq E$ an open set. $f : U \to F$. We say f is **differentiable** at $x_0 \in U$ iff $\exists$ a continuous linear map $L_{x_0} : E \to F$ such that

   $$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - L_{x_0}(h)\|_F}{\|h\|_E} = 0.$$

   (b) If $U \subseteq \mathbb{R}^n, f : U \to \mathbb{R}$, then we say a **partial derivative** of f at x_0 exists in the direction e_i iff

   $$\lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

   exists. Define $\frac{\partial f}{\partial x_i}(x_0) = \partial_i f(x_0)$

   (c) To say that $P = (P_1, P_2, \ldots, P_n)$ is a **partition** of R means that for each $1 \le i \le n, P_i = \{x_j^i\}_{j=0}^{m_i}$ is a partition of $[a_i, b_i]$, and $a_i = x_0^i < x_1^i < \cdots < x_{m_i}^i = b_i$

   (d) To say that P' is a **refinement of P** means

   i. P' is a partition of R

   ii. for $1 \le i \le n$, P' is a refinement of P_i in $\mathbb{R}$

   and, If P' is a refinement of P then

   i. $S' \in \mathcal{R}(P') \Rightarrow S' \subseteq S$ for some $S \in \mathcal{R}(P)$, where $\mathcal{R}$ is the set of all subrectangles.

   ii. $S \in \mathcal{R}(P)$ implies that *every element* of S is in some subrectangle $S' \in \mathcal{R}(P')$ such that $S' \subseteq S$

   (e) Let $A \subseteq \mathbb{R}^n$ be bounded and $f : A \to \mathbb{R}$, a *bounded function*. Let R be a rectangle s.t $A \subseteq R$ and define f on $R \backslash A$ by $f(x) = 0$. Now $f : R \to \mathbb{R}$. Let P be any partition of R. The **upper sum over a partition** is defined as

   $$U(f, P) = \sum_{S \in \mathcal{R}(P)} \sup_{S^\circ} f \ \mathrm{vol}(S), \qquad S^\circ \text{ an open rectangle}$$

   The **lower sum over a partition** is defined as

   $$L(f, P) = \sum_{S \in \mathcal{R}(P)} \inf_{S^\circ} f \ \mathrm{vol}(S)$$

   The **lower integral** is defined as

   $$\underline{\int}_R f = \sup\{L(f, Q) \mid Q \text{ a partition of } R\}$$

   The **upper integral** is defined as

   $$\overline{\int}_R f = \inf\{U(f, P) \mid P \text{ a partition of } R\}$$

   A function f is integrable iff $\overline{\int}_R f = \underline{\int}_R f$.

2. Assume $U \subseteq \mathbb{R}^n$ is open and $f \in C^{(3)}(U)$.

   (a) If f has a local minimum at x_0 then $D_{x_0}f = 0$ and $D^2_{x_0}f(h,h) \geq 0 \; \forall h \in \mathbb{R}^n$.

   (b) If $D_{x_0}f(h) = 0$ and $D^2_{x_0}f(h,h) > 0 \; \forall h \in \mathbb{R}^n \backslash \{0\}$, then f has a local minimum at x_0.

Proof. Proof of (b): The map $u \to D^2_{x_0}f(u,u) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0)u_i u_j$ is continuous and there exists a u_0 such that $\|u_0\| = 1$ and $D^2_{x_0}f(u_0,u_0) \leq D^2_{x_0}f(u,u) \; \forall u \ni \|u\| = 1$. We know that $D^2_{x_0}f(u_0,u_0) > 0$. We show that f has a local minimum at x_0. Choose $\delta > 0$ such that $D_\delta(x_0) \subseteq U$ and $\|h\| < \delta \Rightarrow$

$$f(x_0+h) = f(x_0) + \underbrace{D_{x_0}f(h)}_{=0} + \frac{1}{2}D^2_{x_0}f(h,h) + R_2(x_0,h), \quad \exists M \in \mathbb{R} \ni |R_2(x_0,h)| \leq \frac{1}{6}M\|h\|^3, M > 0$$

Then, by rearranging terms and factoring out and dividing by $|h|^2$

$$\left| \frac{f(x_0+h) - f(x_0) - \frac{1}{2}D^2_{x_0}f(h,h)}{\|h\|^2} \right| < \frac{1}{6}M\|h\|$$

$$\left| \frac{f(x_0+h) - f(x_0)}{\|h\|^2} - \frac{1}{2}D^2_{x_0}f\left(\frac{h}{\|h\|}, \frac{h}{\|h\|}\right) \right| < \frac{1}{6}M\|h\|$$

Let $\varepsilon = \frac{\frac{1}{2}D^2_{x_0}f(u_0,u_0)}{2} > 0$. Choose $\delta > \delta_1 > \|h\| \Rightarrow \frac{1}{6}M\|h\| < \varepsilon$.

$$\left| \frac{f(x_0+h) - f(x_0)}{\|h\|^2} - \frac{1}{2}D^2_{x_0}f\left(\frac{h}{\|h\|}, \frac{h}{\|h\|}\right) \right| < \frac{1}{6}M\|h\| < \varepsilon$$

$$\underbrace{\frac{1}{2}Dx_0 f\,(u_0,u_0) - \varepsilon}_{2\varepsilon - \varepsilon} \leq \frac{1}{2}D^2_{x_0}f\left(\frac{h}{\|h\|}, \frac{h}{\|h\|}\right) - \varepsilon < \frac{f(x_0+h) - f(x_0)}{\|h\|^2}$$

$$2\varepsilon - \varepsilon = \varepsilon < \frac{f(x_0+h) - f(x_0)}{\|h\|^2}$$

So

$$\frac{f(x_0+h) - f(x_0)}{\|h\|^2} > 0$$

Therefore $f(x_0+h) - f(x_0) > 0 \Rightarrow f(x_0) < f(x_0+h) \; \forall h$ such that $0 < \|h\| < \delta_1$, where $x_0 + h \in D_\delta(x_0)$. Therefore x_0 is a minimum! □

3. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$. Then f is differentiable at $x_0 \in U$ iff $\forall 1 \leq i \leq m$ f_i is differentiable at x_0. Here, $f_i = \pi_i \circ f$, where $\pi_i : \mathbb{R}^m \to \mathbb{R}$ defined by $\pi_i(y_i) = y_i$ (picks out the ith coordinate of f).

Proof. ($\Rightarrow$) Assume f is differentiable at $x_0 \in U$. Recall that $|y_i| \leq \sqrt{y_1^2 + y_2^2 + \cdots + y_m^2} = \|y\|_2$, so

$|\pi_i(y)| \leq \|y\|$. Since f is differentiable, we have:

$$0 \leq \left|\pi_i\left(\frac{f(x_0+h)-f(x_0)-Df(x_0)(h)}{\|h\|}\right)\right| \leq \left\|\frac{f(x_0+h)-f(x_0)-Df(x_0)(h)}{\|h\|}\right\|$$
$$0 \leq \left|\frac{\pi_i(f(x_0+h))-\pi_i(f(x_0))-\pi_i(Df(x_0)(h))}{\|h\|}\right| \leq \frac{\|f(x_0+h)-f(x_0)-Df(x_0)(h)\|}{\|h\|}$$
$$0 \leq \frac{|f_i(x_0+h)-f_i(x_0)-\pi_i(Df(x_0)(h))|}{\|h\|} \leq \underbrace{\frac{\|f(x_0+h)-f(x_0)-Df(x_0)(h)\|}{\|h\|}}_{\lim_{h\to 0}=0}$$

The limit of the LHS exists and is zero since the limit of the RHS is zero. Therefore, $Df_i(x_0)$ exists and is $\pi_i \circ Df(x_0)(h) \Rightarrow Df(x_0)(h) = (\pi_1 Df(x_0)(h), \ldots, \pi_m Df(x_0)(h)) = (Df_1(x_0)(h), \ldots, Df_m(x_0)(h))$.

Converse $(\Leftarrow)$ Assume f_i is differentiable at x_0 for each $1 \leq i \leq m$. Therefore $\forall \varepsilon > 0, \forall 1 \leq i \leq m,\ \exists \delta_i > 0$ such that $0 \leq \|h\| \leq \delta_i$. Then have

$$\frac{|f_i(x_0+h)-f_i(x_0)-Df_i(x_0)(h)|}{\|h\|} < \frac{\varepsilon}{\sqrt{m}}$$

Let $\delta = \min(\delta_1, \ldots, \delta_m) > 0$. Choose h such that $0 \leq \|h\| < \delta$. Then...

$$\sum_{i=1}^{m}\left[\frac{f_i(x_0+h)-f_i(x_0)-Df_i(x_0)(h)}{\|h\|}\right]^2 \leq \sum_{i=1}^{m}\left(\frac{\varepsilon^2}{m}\right) = \varepsilon^2$$
$$\frac{1}{\|h\|^2}\cdot\sum_{i=1}^{m}\big(f_i(x_0+h)-f_i(x_0)-Df_i(x_0)(h)\big)^2 \leq \varepsilon^2$$

Take the square root to turn it into the 2-norm...

$$\frac{\big\|f(x_0+h)-f(x_0)-[Df_1(x_0)(h), \ldots, Df_m(x_0)(h)]\big\|}{\|h\|} < \varepsilon$$

$\therefore f$ is differentiable at x_0 and $Df(x_0)(h) = (Df_1(h), \ldots, Df_m(h))$ □

4. Be able to derive the Taylor Series approximation $f(x+h) = f(x) + Df(x)(h) + \frac{1}{2}D^2 f(x)(h,h) + R(x,h)$.

Proof. This proof is for $f \in \mathbb{R}^2$, but is "easily extended". Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}, f \in C^{(3)}$, and the point $(x,y) \in U$. Let $\delta > 0$ s.t $\|(h,k)\| < \frac{\delta}{2} \Rightarrow (x+h, y+k) \in U$. The $\frac{\delta}{2}$ assuredly lets us work with $t = 1$ at the end.

Notice that $\|(x+th, y+tk)-(x,y)\| = \|(th,tk)\| = |t|\|(h,k)\|$. Define $g(t) = f((x,y)+t(h,k)) = f(x+th, y+tk)$. Let $u(t) = (x+th, y+tk)$. Then, taking derivatives, we have (very sloppy notation, but

$h_1 = h, h_2 = k, \vec{h} = (h_1, h_2) \in \mathbb{R}^2$):

$$\begin{aligned} g'(t) &= \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}(u(t))\vec{h_i} = \frac{\partial f}{\partial x}u(t)(h) + \frac{\partial f}{\partial y}u(t)(k) \\ &= D_{u(t)}(f)(\vec{h}) \\ g''(t) &= \sum_{i=1}^{2} \frac{d}{dt}\left[\frac{\partial f}{\partial x_i}u(t)(\vec{h_i})\right] = \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2 f}{x_i x_j}(u(t))(\vec{h_i})(\vec{h_i}) \\ &= \frac{\partial^2 f}{\partial x^2}(u(t))(h^2) + 2\frac{\partial^2 f}{\partial x \partial y}(u(t))(h \cdot k) + \frac{\partial^2 f}{\partial y^2}(u(t))(k^2) \\ &= D^2_{u(t)}(f)(\vec{h}, \vec{h}) \end{aligned}$$

Here $g(t)$ is a function of one variable, so we can just use the one variable Maclaurin/Taylor series, and then plug in.

$$g(t) = g(0) + g'(0)(t) + \frac{1}{2}g''(0)t^2 + \cdots + \frac{1}{k!}g^{(k)}(0) \cdot t^k + R_k$$

So when t=1, and for our $\mathbb{R}^2$ function, substituting in:

$$f(t+h) = f(x) + Df(x)(\vec{h}) + \frac{1}{2}D^2 f(\vec{h}, \vec{h}) + R(x, \vec{h})$$

And whiz-bang, that's it. □

Integration Proofs

1. If P' is a refinement of a partition P of R then $L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)$.

   Proof.

$$\begin{aligned} U(f,P') &= \sum_{S' \in \mathcal{R}(P)} \sup_{S^{\circ\prime}} f \ \text{vol}(S') \\ &= \sum_{S \in \mathcal{R}(P)} \sum_{\substack{S' \in \mathcal{R}(P') \\ S' \subseteq S}} \sup_{S^{\circ\prime}} f \ \text{vol}(S') \leq \sum_{S \in \mathcal{R}(P)} \sum_{\substack{S' \in \mathcal{R}(P) \\ S' \subseteq S}} \sup_{S^{\circ}} f \ \text{vol}(S') \end{aligned}$$

   The RHS (the inequality part) of which continues...

$$\begin{aligned} &= \sum_{S \in \mathcal{R}(P)} \sup_{S^{\circ}} f \sum_{\substack{S' \in \mathcal{R}(P) \\ S' \subseteq S}} \text{vol}(S') \\ &= \sum_{S \in \mathcal{R}(P)} \sup_{S^{\circ}} f \ \text{vol}(S) \\ &= U(f,P) \end{aligned}$$

   $\therefore U(f,P') \leq U(f,P)$ □

2. f is integrable iff $\forall \varepsilon > 0$ if there exists a partition P_ε of R such that $0 \leq U(f,P) - L(f,P) < \varepsilon$ for every refinement P of P_ε

Proof. Let $\varepsilon > 0$. $\overline{\int}_R f + \frac{\varepsilon}{2} > \sup f$. So there exists a partition P' of R s.t $\overline{\int}_R f - \frac{\varepsilon}{2} \geq U(f, P') \Rightarrow \frac{\varepsilon}{2} \geq U(f, P') - \overline{\int}_R f$. Similarly $\underline{\int}_R f - \frac{\varepsilon}{2} < \sup f \Rightarrow \exists$ a partition P'' of R s.t $\underline{\int}_R f - \frac{\varepsilon}{2} \leq L(f, P'')$. Then have

$$\underline{\int}_R f - L(f, P'') \leq \frac{\varepsilon}{2}$$

and

$$U(f, P') - \overline{\int}_R f \leq \frac{\varepsilon}{2}$$

Let $P_\varepsilon = P' \cup P''$, let P be a refinement of P_ε. So if $\overline{\int} = \underline{\int}$ then $U(f, P) - L(f, P) < \varepsilon$. □