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ON THE STABILITY OF HIGH LEWIS NUMBER COMBUSTION FRONTS

Anna Ghazaryan

Department of Mathematics, University of North Carolina at Chapel Hill
Chapel Hill, NC 27599, USA

Christopher K. R. T Jones

Department of Mathematics, University of North Carolina at Chapel Hill
Chapel Hill, NC 27599, USA
and
Warwick Mathematics Institute, University of Warwick
Coventry CV4 7AL, UK

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Abstract. We consider wavefronts that arise in a mathematical model for high Lewis number combustion processes. An efficient method for the proof of the existence and uniqueness of combustion fronts is provided by geometric singular perturbation theory. The fronts supported by the model with very large Lewis numbers are small perturbations of the front supported by the model with infinite Lewis number. The question of stability for the fronts is more complicated. Besides discrete spectrum, the system possesses essential spectrum up to the imaginary axis. We show how a geometric approach which involves construction of the Stability Index Bundles can be used to relate the spectral stability of wavefronts with high Lewis number to the spectral stability of the front in the case of infinite Lewis number. We discuss the implication for nonlinear stability of fronts with high Lewis number. This work builds on the ideas developed by Gardner and Jones [12] and generalized in the papers by Bates, Fife, Gardner and Jones [3, 4].

1. Introduction.

1.1. Model.

We consider a well-known model for the propagation of combustion waves in the case of premixed fuel, with no heat loss, in one spatial dimension $x \in \mathbb{R}$. The system describing evolution of the temperature u and concentration of the fuel y reads

$$\begin{aligned} u_t &= u_{xx} + y\Omega(u), \\ y_t &= \varepsilon y_{xx} - \beta y \Omega(u). \end{aligned} \tag{1}$$

The reaction rate has the form of an Arrhenius law without ignition cut off:

$$\Omega(u) = e^{-1/u} \text{ for } u > 0 \text{ and } \Omega(u) = 0 \text{ otherwise.}$$

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The system has two parameters. One is the exothermicity parameter $\beta > 0$ which is the ratio of the activation energy to the heat of the reaction. The other is the reciprocal of the Lewis number $\varepsilon = 1/\mathrm{Le} > 0$. Therefore, ε represents the ratio of the fuel diffusivity to the heat diffusivity. The system has been studied for various parameter regimes. Of interest to us are traveling wave solutions to (1) in two cases. One is the system (1) with $\varepsilon = 0$ ($\mathrm{Le} = \infty$). Its physical prototype is the combustion of solid fuels, more precisely, combustion that involves the solid phase only with no gaseous products present. The other is the case of $0 < \varepsilon \ll 1$, i.e., when Le is very large but finite. This situation is also physical: (1) then describes burning of very high density fluids at high temperatures. The interest in the transition between zero and nonzero ε is explained by two facts. First is that during the burning of solid fuels some liquefaction of the fuel might occur in the reaction zone, thus causing a non-zero value of $\varepsilon = 1/\mathrm{Le}$. The other is that the model for combustion of solid fuels originated in physics as the limit of (1) as $\varepsilon \to 0$ (see [20] and the references therein). Since the limit is singular it is not obvious at all if the systems with zero and nonzero ε have the same properties. The existence and properties of combustion waves for both cases have been studied thoroughly. To our knowledge, the comparison of the stability properties has been addressed only from a numerical point of view. Our main goal is to use analytical methods to relate the stability of the combustion wave of (1) with $\varepsilon > 0$ to the stability of the combustion wave of (1) with $\varepsilon = 0$.

1.2. **Existence of the front.** We concentrate our attention on the traveling wave solutions of front type. In particular, we are interested in fronts of (1) that asymptotically connect

$$(u, y) = (1/\beta, 0) \text{ at } -\infty, \text{ and } (u, y) = (0, 1) \text{ at } \infty, \tag{2}$$

and approach these rest states at exponential rates. The boundary conditions represent the physical state where the fuel is completely burnt, i.e., $y = 0$, and the maximal temperature $u = \frac{1}{\beta}$ is reached, and the state when none of the fuel is yet burnt and the temperature u is still zero.

To find a traveling wave we introduce in (1) a moving coordinate frame $\xi = x - ct$,

$$\begin{aligned} u_t &= u_{\xi\xi} + cu_\xi + y\Omega(u), \\ y_t &= \varepsilon y_{\xi\xi} + cy_\xi - \beta y\Omega(u). \end{aligned} \tag{3}$$

In the system above, with an abuse of notation, we think of u and y as functions of ξ, not x, t. The parameter c represents the speed of the wave. Traveling waves are sought as stationary solutions of (3),

$$\begin{aligned} u'' + cu' &= -y\Omega(u), \\ \varepsilon y'' + cy' &= \beta y\Omega(u), \end{aligned} \tag{4}$$

where the derivative is with respect to ξ.

There is a linear algebraic relation satisfied along any solution of (4), coming from an invariant of the equations,

$$\beta u'' + \beta cu' + \varepsilon y'' + cy' = 0.$$

Thus, the quantity $\beta u' + \beta cu + \varepsilon y' + cy$ is conserved along trajectories. To ensure the boundary conditions (2) are satisfied, it must equal to c:

$$\beta u' + \beta cu + \varepsilon y' + cy = c. \tag{5}$$

It is easy to see from (5), that, if $0 \le \varepsilon < 1$, then there are no standing waves ($c = 0$) satisfying (2). We fix the direction of propagation of the front by choosing $c > 0$, i.e. we choose to look for fronts that leave the burned state behind. It is also known (see Sect. 4 for references) that both u and y that solve the traveling wave equation (4) are positive and monotone: u is decreasing and y is increasing.

For (4) with no additional assumptions on $\beta > 0$, and when $\varepsilon = 0$, the existence and uniqueness of a front that converges to both of its rest states exponentially fast has been proved in [7, 26]. There are also solutions with algebraic rates of decay, but these are considered to be of little interest [26]. The front has also been observed numerically in [27].

In the case of $0 < \varepsilon \ll 1$ to prove the existence and uniqueness of the front, geometric singular perturbation theory seems to be a natural approach. More precisely, the system has a slow-fast structure. In the case of $\varepsilon = 0$ the flow is restricted to a two dimensional invariant manifold. The manifold is normally hyperbolic and attracting, therefore, by Fenichel's First theorem [11], it perturbs to an attracting manifold invariant for the flow with $\varepsilon > 0$. For the reduced problem, the lower dimension of the problem can be used to show that the front in the $\varepsilon = 0$ case is realized as a transversal intersection of relevant invariant manifolds. Transversality can be proved by a Melnikov integral calculation [2], or by following the blueprint provided by the proof of the existence and uniqueness of subsonic detonation waves [14]. The proof follows the same logic as the proof in [12, 4], where the existence and uniqueness of a traveling wave was proved for an equation and a system arising in a phase field and a generalized phase field models. In addition, in [2] the formula for the corrective term for the velocity of propagation of the perturbed front has also been obtained.

An essentially different approach based on Leray-Schauder degree theory has been used in [6] to prove the existence and uniqueness of the front for when $0 < \varepsilon < 1$, and in [20] for $\varepsilon = 0$.

We state the results on the existence of the waves for small ε discussed above in the following theorem.

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that for each $0 \le \varepsilon \le \varepsilon_0$ system (1) has a unique, up to translation, traveling wave that satisfies boundary conditions (2) and does so at exponential rates. The speed $c = c(\varepsilon)$ of the wave is a continuous function of $\varepsilon \in [0, \varepsilon_0]$. As sets in the phase space, the orbits corresponding to $0 < \varepsilon < \varepsilon_0$ as $\varepsilon \to 0$ converge to the unique orbit of the system with $\varepsilon = 0$.*

1.3. **Stability.** Concerning the stability properties of the combustion front, the question we want to address here is about the relationship between two cases: $\varepsilon = 0$ and $\varepsilon > 0$. More precisely, do combustion fronts with $0 < \varepsilon \ll 1$ inherit stability properties of the front with $\varepsilon = 0$?

We denote by (u_f, y_f) the front, and by c_f the corresponding value of its speed, i.e., (u_f, y_f), c_f refers to $(u_0(\xi), y_0(\xi))$, c_0, $\xi = x - c_0 t$, when $\varepsilon = 0$, and $(u_\varepsilon(\xi), y_\varepsilon(\xi))$, c_ε, $\xi = x - c_\varepsilon t$, when $\varepsilon > 0$. The linearization L of the right hand side of (3) with $c = c_f$ about (u_f, y_f) is given by

$$L \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi + y_f \Omega_u(u_f) & \Omega(u_f) \\ -\beta y_f \Omega_u(u_f) & \varepsilon\partial_{\xi\xi} + c\partial_\xi - \beta\Omega(u_f) \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix},$$

where p and r are functions of ξ. The stability of the front is determined by the spectrum of L. The eigenvalue problem for L reads

$$\begin{aligned} \lambda p &= p_{\xi\xi} + cp_\xi + \Omega(u_f)r + y_f\Omega_u(u_f)p, \\ \lambda r &= \varepsilon r_{\xi\xi} + cr_\xi - \beta\Omega(u_f)r - \beta y_f\Omega_u(u_f)p. \end{aligned} \tag{6}$$

A traveling wave is called spectrally stable if the spectrum of the linearization of the system about the traveling wave is contained in the left half-plane of the complex plane, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$. Generally speaking, the spectral stability need not imply the linear stability of the traveling wave, i.e., the decay of the solutions of the linearization of PDE about the traveling wave (with the exception of the single mode due to the translation invariance).

If the linearized operator is sectorial, the linear stability is guaranteed [17, Sect. 5.1] if there exists $B > 0$ such that the spectrum belongs to the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -B\}$ with the exception of a simple eigenvalue at zero, caused by translation symmetry. In Section 2.1, we show that the essential spectrum of the linearization of (1) about the front reaches the imaginary axis for any $\varepsilon \geq 0$. Nevertheless, for this problem, it is possible to shift the essential spectrum to the open left half-plane using exponential weights. The translational eigenvalue, $\lambda = 0$, is simple in the case $\varepsilon = 0$, as it has been shown analytically in [15] using Evans function analysis. We will show that it is simple in the case of $0 < \varepsilon \ll 1$ as well. The simplicity of $\lambda = 0$ is also suggested by a numerical investigation, for both $\varepsilon = 0$ and nonzero, see [2, 16, 25]. Analytic results about location of the discrete spectrum for this model do not exist for either zero or nonzero ε.

For the $\varepsilon > 0$ case it has been shown by numerical methods in [16] that there exist parameter regimes where there are no isolated eigenvalues in the closed right half-plane other than zero. In this case the front is spectrally stable in some exponentially weighted space. For fixed $\varepsilon > 0$ the linearized operator is the perturbation of the Laplacian by lower order derivatives and bounded operators, and, thus, it is also sectorial [17, 24]. Therefore, the linear stability of the front in appropriate exponentially weighted spaces follows from the spectral stability. The nonlinear stability does not simply follow from the linear stability because of the insufficient smoothness of the nonlinearity in the weighted space. Nevertheless, this issue is resolved in a forthcoming paper [13], where it is shown that the front is nonlinearly stable in the weighted norm against perturbations that are sufficiently small both in the weighted H^1 norm and the regular H^1 norm without a weight.

A similar nonlinear stability result has been obtained in [15] for the case $\varepsilon = 0$, again if a parameter regime is assumed for which there are no unstable eigenvalues. For this case one of the main difficulties in the stability analysis is that the linearized operator is not sectorial. The essential spectrum contains the imaginary axis. Exponential weights with positive rates shift the essential spectrum to the left of the imaginary axis, but the linearized operator generates not an analytic but a C^0 semigroup. Nevertheless, semigroup estimates were obtained in [15] that show that with respect to a certain exponentially weighted norm on the linear level perturbations decay in time exponentially. The approach to nonlinear stability in [15] is similar to the one in [13]. The combination of estimates for the perturbations in the norms with and without weight is used to show that the nonlinearity is dominated by linear terms.

To discuss the point spectrum and its robustness under perturbations, we use the concept of the Evans function. The Evans function is an analytic function of a complex variable λ, defined for λ to the right of the essential spectrum [1]. Zeroes of the Evans function coincide with the eigenvalues of the linearized operator with the order of a zero being equal to the multiplicity of the eigenvalue. In Section 2.2, using the same approach as in [19], we show that the Evans function for our system can be analytically continued across the boundary of the essential spectrum. We will call this analytic continuation of the Evans function the extended Evans function. Embedded in the essential spectrum, eigenvalues are still zeroes of the extended Evans function, but not necessarily vice versa.

For both cases, $\varepsilon = 0$ and $\varepsilon > 0$, we distinguish three situations.

A. There is at least one zero of the Evans function, i.e., isolated eigenvalue, in the open right half plane of the complex plane: the front is truly unstable.
B. There are no zeroes of the Evans function to the right of the imaginary axis. Moreover, with the exception of the simple zero at the origin, the extended Evans function has no zeroes on the imaginary axis: the most stable case.
C. There are no zeroes of the Evans function to the right of the imaginary axis, but there are zeroes on the imaginary axis in addition to the zero at the origin: marginal case.

The main result of this paper is that cases A and B with $\varepsilon = 0$ are robust under singular perturbations with $\varepsilon > 0$ sufficiently small. For brevity, in what follows, we denote the extended Evans function E_0 when $\varepsilon = 0$, and E_ε when $\varepsilon > 0$.

Theorem 1.2. *If the front (u_0, y_0) is unstable then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the front $(u_\varepsilon, y_\varepsilon)$ is also unstable.*

Theorem 1.3. *Assume the front (u_0, y_0) is spectrally stable, and, moreover, the statement B about E_0 is true. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the statement B is true for E_ε; therefore, the front $(u_\varepsilon, y_\varepsilon)$ is also spectrally stable.*

Theorems 1.2 and 1.3 are consequences of the following key proposition about the zeroes of the Evans function.

Proposition 1. *Assume that λ_0 is a zero of order m of E_0. There exists $\varepsilon_0 > 0$ such that for any ε ($0 < \varepsilon < \varepsilon_0$), in a neighborhood of λ_0 of order ε, there are exactly m zeroes (counting multiplicity) of E_ε.*

This statement is proved in Section 3. These analytic results confirm the expectations based on the numerical analysis in [2] where a count of the number of eigenvalues in various bounded domains of the complex plane has been performed and compared for $\varepsilon = 0$ and $\varepsilon > 0$. Here we have also used the fact that $\lambda = 0$ is a simple eigenvalue when $\varepsilon = 0$, and, because of Proposition 1, persists as a simple eigenvalue when $\varepsilon > 0$. Therefore no eigenvalue can cross into the right half-plane through the origin when the problem is perturbed by introducing non-zero ε.

There is numerical evidence [16] that the stability of the front ($0 \leq \varepsilon \ll 1$) depends on the parameter β. It has been shown numerically that there exist $\beta_0 = O(1)$ such that for $\beta < \beta_0$ the linearization about the front does not possess any unstable eigenvalues. For larger values of β a pair of complex conjugate eigenvalues crosses the imaginary axis from left to right, causing a so-called pulsating instability. The occurrence of Hopf bifurcation, with the speed of the front as the bifurcation parameter, has also been noted in [25].

The numerical observation of pulsating instability is not unexpected. Its presence has been proved under the assumption of high energy activation [22]. In this situation, the existence of two complex conjugate, purely imaginary eigenvalues has been obtained. But in some way, the behavior of the unstable eigenvalues as functions of β, when β is large, has not been completely captured by the numerics. In other words, it is not clear from the numerics what happens with the unstable eigenvalues for $\beta \gg 1$. Based on our estimates in Sect. 4.1, we claim that with further increase of β, at some point, the real parts of the unstable eigenvalues start to decrease. The eigenvalues approach the imaginary axis. This observation is in agreement with the stability analysis performed in the high energy activation limit in [22]. The numerical investigation of bifurcations as large β varies will be addressed in a forthcoming paper.

1.4. **Main technique: Stability Index Bundles.** To prove our results we apply the technique developed in [1], where the stability index was introduced. The stability index is a topological invariant which counts the number of eigenvalues inside a given closed contour. If there exists a contour K enclosing all of the unstable eigenvalues, then the spectral stability of the wave can be concluded from the stability index of the contour. More precisely, this topological invariant is the first Chern number of the Stability Index Bundle. The Stability Index Bundle, also called the augmented unstable bundle [1], is a bundle with fibers formed by certain invariant manifolds in the phase space of the linearization (6) and the base given by a compactification of an infinite cylinder $\mathbb{R} \times K$ (for the real space variable and the complex parameter λ) capped at $\pm\infty$ by the contour K combined with its interior.

The first Chern number of the Stability Index Bundle coincides with the winding number of Evans function over the contour K and therefore counts eigenvalues inside K, because they are zeroes of the Evans function. The detailed construction of the bundle is presented in Sect. 3 and the description of the contour K is discussed in Sect. 2.2.

Because of the slow-fast structure of the eigenvalue problem (6), the Stability Index Bundle can be decomposed into the Whitney sum of the associated slow and fast subbundles [23]. In this particular case we were able to prove that the slow bundle is, in fact, the full unstable bundle. The topological nature of the stability index makes it robust under small perturbations. Therefore the Chern number of the full system ($0 < \varepsilon \ll 1$) is equal to the Chern number of the reduced system ($\varepsilon = 0$) and the statement of the theorem follows. The construction of the Stability Index Bundle is performed in the framework of exterior powers $\Lambda^k(\mathbb{C})$ (see [10, 28]) of $\mathbb{C}^4$.

The proof of the spectral result is close in spirit to the topological approach in the stability analysis in [12] where the Stability Index Bundle was used to identify the spectral properties of traveling waves appearing in a phase field model. The phase field model in [12] is given by one partial differential equation, but of higher (6th) order with even derivatives only, and is a singular perturbation of a second order equation. For the traveling wave in the phase field, unlike our case, both slow and fast bundles are present, but only the slow bundle contributes to the spectrum. The stability analysis of the front in a phase field model for hypercooled solidification [3] also closely follows the stability proof given in [12].

The plan of the paper is as follows. In Sect. 2 we construct the extended Evans function. To do so we need to know the location of the essential spectrum. The Stability Index Bundle is defined by its fibers and the base. In Sect. 3 we define the fibers. The base of the bundle is defined in Sect. 4 by means of a construction of a contour in the complex plane that leads us to the proof of Theorems 1.2 and 1.3. We call such contour the index contour. The existence of the index contour is guaranteed by the bound on the moduli of the unstable eigenvalues, and properties of zeroes of an analytic function, here, the extended Evans function. Monotonicity of the front (u_f, u_f) is used to obtain an estimate on the real parts of the unstable eigenvalues.

2. **Evans Function.** The Evans function for (1) was successfully used to study the discrete spectrum of the problem with fixed ε [2, 15, 16, 27]. We use the Evans function to construct the base of the Stability Index Bundle. To encircle the translational eigenvalue at the origin, embedded in the essential spectrum, in the consideration, we need to analytically continue the Evans function across the right boundary of the essential spectrum. This section consists of two parts. First, we find the location of the essential spectrum, then define the extended Evans function.

2.1. **Essential spectrum.** In this section we find the location of the essential spectrum, which is the complement of the point spectrum in the spectrum.

Case $\varepsilon > 0$. The eigenvalue problem (6), written as a first order ODE, on the fast scale $\eta = \xi/\varepsilon$ reads

$$\begin{aligned}
\dot{p} &= \varepsilon q, \\
\dot{q} &= \varepsilon(-cq - \Omega(u_f)r - y_f\Omega_u(u_f)p + \lambda p), \\
\dot{r} &= \varepsilon s, \\
\dot{s} &= -cs + \beta\Omega(u_f)r + \beta y_f\Omega_u(u_f)p + \lambda r.
\end{aligned} \tag{7}$$

The right-hand side of this system is an action of the matrix

$$M(\xi,\lambda,\varepsilon) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon(\lambda - y_f\Omega_u(u_f)) & -c\varepsilon & -\varepsilon\Omega(u_f) & 0 \\ 0 & 0 & 0 & \varepsilon \\ \beta y_f\Omega_u(u_f) & 0 & \lambda + \beta\Omega(u_f) & -c \end{pmatrix}$$

on the vector $(u, v, y, z)^T$. Let $M^{\pm}(\lambda,\varepsilon) = \lim_{\xi\to\pm\infty} M(\xi,\lambda,\varepsilon)$. Then

$$M^-(\lambda,\varepsilon) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon\lambda & -c\varepsilon & -e^{-\beta}\varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & \lambda + \beta e^{-\beta} & -c \end{pmatrix}, \quad M^+(\lambda,\varepsilon) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon\lambda & -c\varepsilon & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & \lambda & -c \end{pmatrix}.$$

The eigenvalues of $M^{\pm}(\lambda,\varepsilon)$ are called the spatial eigenvalues as opposed to the temporal eigenvalues λ. The eigenvalues of $M^-(\lambda,\varepsilon)$ are

$$\begin{aligned}
\kappa_1^- &= \tfrac{1}{2}(-c - \sqrt{c^2 + 4\varepsilon(\lambda + \beta e^{-\beta})}), & \kappa_3^- &= \frac{\varepsilon}{2}(-c - \sqrt{c^2 + 4\lambda}), \\
\kappa_2^- &= \tfrac{1}{2}(-c + \sqrt{c^2 + 4\varepsilon(\lambda + \beta e^{-\beta})}), & \kappa_4^- &= \frac{\varepsilon}{2}(-c + \sqrt{c^2 + 4\lambda}).
\end{aligned}$$

When λ crosses the imaginary axis from right to left, κ_4^- crosses the imaginary axis from right to left. The boundaries of the essential spectrum due to the behavior at $+\infty$ are curves

$$\{\lambda = -\varepsilon\nu^2 + ci\nu;\ \nu \in \mathbb{R}\} \cup \{\lambda = -\nu^2 + ci\nu;\ \nu \in \mathbb{R}\}. \tag{8}$$

The eigenvalues of $M^+(\lambda,\varepsilon)$ are

$$\begin{aligned}
\kappa_1^+ &= \tfrac{1}{2}(-c - \sqrt{c^2 + 4\varepsilon\lambda}), & \kappa_3^+ &= \frac{\varepsilon}{2}(-c - \sqrt{c^2 + 4\lambda}), \\
\kappa_2^+ &= \tfrac{1}{2}(-c + \sqrt{c^2 + 4\varepsilon\lambda}), & \kappa_4^+ &= \frac{\varepsilon}{2}(-c + \sqrt{c^2 + 4\lambda}).
\end{aligned}$$

When λ crosses the imaginary axis from right to left, both eigenvalues with positive real parts, κ_2^+, κ_4^+, cross the imaginary axis from right to left, and coincide at $\lambda = 0$. The boundaries of the essential spectrum due to the behavior at $-\infty$ are given by the curves

$$\{\lambda = -\varepsilon\nu^2 + ci\nu - \beta e^{-\beta};\ \nu \in \mathbb{R}\} \cup \{\lambda = -\nu^2 + ci\nu;\ \nu \in \mathbb{R}\}. \tag{9}$$

The set of curves (9)-(8) divides the complex plane into regions which either are completely covered by spectrum or, otherwise, contain only discrete eigenvalues [17]. There is a component of it which contains the open right half-plane of the complex plane. From the estimates described in Sect. 4.1, we know that there can be no eigenvalues with large positive real parts. Thus this region cannot be covered entirely by eigenvalues, i.e. does not contain essential spectrum. Therefore the region to the right of the rightmost parabola from (8) and (9)

$$\{\lambda = -\varepsilon\nu^2 + ci\nu;\ \nu \in \mathbb{R}\} \tag{10}$$

contains only discrete spectrum, i.e. isolated eigenvalues of finite multiplicity. The essential spectrum is bounded on the right by (10) and includes that curve (see Fig. 1).

Case $\varepsilon = 0$. We rewrite the eigenvalue problem (6) as a first order ODE:

$$\begin{aligned} p' &= q, \\ q' &= -cq - \Omega(u_f)r - y_f\Omega_u(u_f)p + \lambda p, \\ r' &= \frac{\beta}{c}\Omega(u_f)r + \frac{\beta}{c}y_f\Omega_u(u_f)p + \frac{\lambda}{c}r. \end{aligned}$$

The right-hand side of the last system is an action of the matrix

$$M(\xi,\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - y_f\Omega_u(u_f) & -c & -\Omega(u_f) \\ \frac{\beta}{c}y_f\Omega_u(u_f) & 0 & \frac{1}{c}(\lambda + \beta\Omega(u_f)) \end{pmatrix}$$

on the vector $(u,v,y)^T$. We denote $M^{\pm}(\lambda) = \lim_{\xi\to\pm\infty} M(\xi,\lambda)$:

$$M^-(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & -c & -e^{-\beta} \\ 0 & 0 & \frac{\lambda}{c} + \frac{\beta}{c}e^{-\beta} \end{pmatrix}, \qquad M^+(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & -c & 0 \\ 0 & 0 & \frac{\lambda}{c} \end{pmatrix}.$$

The eigenvalues of $M^-(\lambda)$ are

$$\kappa_1^- = \frac{1}{c}(\lambda + \beta e^{-\beta}), \quad \kappa_2^- = \frac{1}{2}(-c - \sqrt{c^2+4\lambda}), \quad \kappa_3^- = \frac{1}{2}(-c + \sqrt{c^2+4\lambda}).$$

For fixed $\beta > 0$, if $\operatorname{Re}\lambda > 0$, then $\operatorname{Re}\kappa_1^-$, $\kappa_3^- > 0$, $\operatorname{Re}\kappa_2^- < 0$. When λ crosses the imaginary axis from right to left κ_3^- crosses the imaginary axis from right to left.

The eigenvalues of $M^+(\lambda)$ are

$$\kappa_1^+ = \frac{\lambda}{c}, \quad \kappa_2^+ = \frac{1}{2}(-c - \sqrt{c^2+4\lambda}), \quad \kappa_3^+ = \frac{1}{2}(-c + \sqrt{c^2+4\lambda}).$$

If $\operatorname{Re}\lambda > 0$, then $\operatorname{Re}\kappa_1^+$, $\kappa_3^+ > 0$, $\operatorname{Re}\kappa_2^+ < 0$. When λ crosses the imaginary axis from right to left both eigenvalues with positive real parts, κ_1^+, κ_3^+, cross the imaginary axis from right to left. The boundaries of the essential spectrum due to the behavior at $+\infty$ are curves

$$\{\lambda = ci\nu;\ \nu \in \mathbb{R}\} \cup \{\lambda = -\nu^2 + ci\nu;\ \nu \in \mathbb{R}\}. \tag{11}$$

The boundaries of the essential spectrum due to the behavior at $-\infty$ are given by the curves

$$\{\lambda = ci\nu - \beta e^{-\beta};\ \nu \in \mathbb{R}\} \cup \{\lambda = -\nu^2 + ci\nu;\ \nu \in \mathbb{R}\}. \tag{12}$$

The rightmost curve $\{\lambda = ci\nu;\ \nu \in \mathbb{R}\}$ in (11) and (12) is the imaginary axis. The same argument as in the case of $\varepsilon = 0$ shows that the essential spectrum belongs to the region to the left of and on the imaginary axis (see Fig. 1). The open right half plane can contain isolated eigenvalues of finite multiplicity only.

Analyzing the linear dispersion relations $d^{\pm}(\lambda, \nu) = 0$ at the asymptotic states at $\pm\infty$ that are relations between temporal eigenvalues λ and the eigenvalues of $M^{\pm}$, we see that the system on the linear level transports perturbations in the direction opposite to the propagation of the front, i. e., to $-\infty$. Indeed, that follows from the sign of the group velocity that is easily calculated: $c_{gr} = -\frac{\mathrm{d}\,\mathrm{Im}\,\lambda}{\mathrm{d}\,\nu} = -c < 0$.

The way the spatial eigenvalues cross the imaginary axis is important for the construction of the Evans function (see the next section).

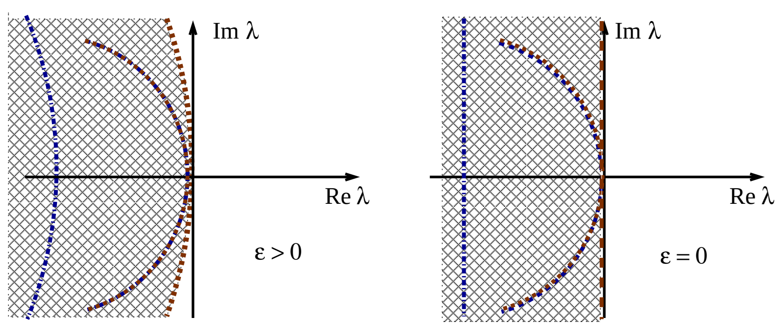

FIGURE 1. The essential spectrum belongs to the region to the left of the rightmost curve and contains that curve.

2.2. **The Evans function and its analytic continuation.** The Evans function is defined, and analytic, for λ's to the right of the essential spectrum. The discrete isolated eigenvalues of the linearized operator are zeroes of the Evans function [1].

In certain situations the Evans function can be analytically continued across the boundary of the essential spectrum as well. In particular, it is possible if the spatial eigenvalues cross the imaginary axis on the λ plane moving in the same direction when λ crosses the imaginary axis from right to left. In the case when a unique eigenvalue with positive real part moves into the essential spectrum as λ crosses the imaginary axis, the analytic continuation of the Evans function has been constructed in [18]. In our situation the construction is similar.

First, we construct the extended Evans function in the case $\varepsilon > 0$. We know that for $\mathrm{Re}\,\lambda > 0$, both $M_{\pm}(\lambda, \varepsilon)$ have two eigenvalues of positive real part (let us call these sets $\sigma^u_{\pm}(\lambda)$, respectively) and two eigenvalues of negative real part ($\sigma^s_{\pm}(\lambda)$). By substituting $\eta = \frac{1}{2k}\ln\left(\frac{1+\tau}{1-\tau}\right)$, where $k > 0$ is a constant, in the eigenvalue problem (7) we obtain the autonomous system

$$\begin{aligned} \dot{P} &= M(\tau, \lambda, \varepsilon)P, \\ \dot{\tau} &= k(1 - \tau^2), \end{aligned} \tag{13}$$

where $P = (p, q, r, s)$ and differentiation is with respect to η again. Since $M(\tau, \lambda, \varepsilon)$ approaches $M_\pm(\lambda, \varepsilon)$ exponentially fast then, for an appropriately chosen k, equation (13) is C^1 on $\mathbb{C}^4 \times [-1, 1]$ (see [1]). We will consider the equation generated by (13) on $\Lambda^2\mathbb{C}^4 \times [-1, 1]$,

$$\begin{aligned} \dot{\hat{P}} &= M^{(2)}(\tau, \lambda, \varepsilon)\hat{P}, \\ \dot{\tau} &= k(1 - \tau^2). \end{aligned} \tag{14}$$

If P_1, $P_2 \in \mathbb{C}^4$ are solutions of (13), then $\hat{P} = P_1 \wedge P_2 \in \Lambda^2\mathbb{C}^4$ solves (14), where $M^{(2)}$ is defined as $M^{(2)}\hat{P} = (MP_1) \wedge P_2 + P_1 \wedge (MP_2)$ (see [8]). The eigenvalues of the asymptotic systems $\dot{\hat{P}} = M^{(2)}_\pm(\lambda, \varepsilon)\hat{P}$ are the sums of any pair of the eigenvalues of $M_\pm$ (see [5]). If $\operatorname{Re}\lambda > 0$, we set $\alpha_-(\lambda)$ to be the eigenvalue of $M^{(2)}_-(\lambda, \varepsilon)$ with the largest real part, and $\alpha_+(\lambda)$ to be the eigenvalue of $M^{(2)}_+(\lambda, \varepsilon)$ with the smallest negative real part. More precisely, we set $\alpha_-(\lambda) = \kappa_2^- + \kappa_4^-$ and $\alpha_+(\lambda) = \kappa_2^+ + \kappa_4^+$.

Both $\alpha_-(\lambda)$ and $\alpha_+(\lambda)$ are well defined not only for λ with positive real parts but also for λ with $\operatorname{Re}\lambda > -\gamma$ for some $\gamma < \frac{c^2}{4}$ and analytic in λ. If, additionally, $\gamma < \beta e^{-\beta}$ then both $\alpha_\pm(\lambda)$ are simple. Therefore the associated eigenvectors depend on λ analytically. For $\operatorname{Re}\lambda > 0$ then the Evans function can be constructed in the same way as in [1, Sect. 4] and continued analytically in the region $\operatorname{Re}\lambda > -\gamma$.

The Evans function for the case $\varepsilon = 0$ is defined in a similar fashion.

3. **Stability Index Bundle.** In this key section we construct the Stability Index Bundle: a bundle whose topological properties can be used to locate eigenvalues of the linearized problem. We start by using the slow-fast structure of the eigenvalue problem (6) to obtain a description for the flow that depends on ε $(0 \le \varepsilon \ll 1)$ continuously. Such a description allows us to define the fibers of the Stability Index Bundle in exactly the same way as in [1]. Moreover, it will follow from the construction that the fibers of the Stability Index are determined only by the slow dynamics of the flow. We use this information combined with estimates on the moduli of the unstable eigenvalues to choose an appropriate base for the bundle.

3.1. **Fibers.** The eigenvalue problem for the linearization about the front reads

$$\begin{aligned} p' &= q, \\ q' &= -cq + (\lambda - y_f\Omega_u(u_f))p - \Omega(u_f)r, \\ r' &= s, \\ \varepsilon s' &= \beta y_f\Omega_u(u_f)p + (\beta\Omega(u_f) + \lambda)r - cs. \end{aligned} \tag{15}$$

Again, as in the definition of the Evans function, following [1], we transform (15) into an autonomous system by introducing a new dependent variable τ defined by the relation

$$\xi = \frac{1}{2\kappa}\ln\left(\frac{1+\tau}{1-\tau}\right), \text{ or } \tau(\xi) = \frac{e^{2\kappa\xi} - 1}{e^{2\kappa\xi} + 1},$$

where $\kappa > 0$ is a constant. The extended system is

$$\begin{aligned} \tau' &= \kappa(1 - \tau^2), \\ p' &= q, \\ q' &= -cq + (\lambda - y_f\Omega_u(u_f))p - \Omega(u_f)r, \\ r' &= s, \\ \varepsilon s' &= \beta y_f\Omega_u(u_f)p + (\beta\Omega(u_f) + \lambda)r - cs, \end{aligned}$$

where derivatives are taken with respect to ξ, but now with an abuse of notation y_f and u_f are functions of τ.

The front (y_f, u_f) is converging to its rest states at rates independent of ε on the slow scale. It has been shown in [1] that there exists κ such that (15) is C^1.

In the fast scaling

$$\begin{aligned}
\dot{\tau} &= \varepsilon\kappa(1-\tau^2),\\
\dot{p} &= \varepsilon q,\\
\dot{q} &= \varepsilon(-cq + (\lambda - y_f\Omega_u(u_f))p - \Omega(u_f)r),\\
\dot{r} &= \varepsilon s,\\
\dot{s} &= \beta y_f\Omega_u(u_f)p + (\beta\Omega(u_f)+\lambda)r - cs.
\end{aligned} \tag{16}$$

For brevity, we denote

$$A(\tau,\varepsilon,\lambda) := \begin{pmatrix} 0 & \varepsilon & 0 & 0\\ \varepsilon(\lambda - y_f\Omega_u(u_f)) & -\varepsilon c & -\varepsilon\Omega(u_f) & 0\\ 0 & 0 & 0 & \varepsilon\\ \beta y_f\Omega_u(u_f) & 0 & \beta\Omega(u_f)+\lambda & -c \end{pmatrix}$$

System (16) reads

$$\begin{aligned}
\dot{P} &= A(\tau,\varepsilon,\lambda)P, \quad P = (p,q,r,s)^T,\\
\dot{\tau} &= \varepsilon\kappa(1-\tau^2).
\end{aligned} \tag{17}$$

Here s is the fast variable and p, q, r, τ are the slow variables. When $\varepsilon = 0$, the system (16), and, equivalently, (17) have a set of equilibrium points

$$s = \frac{1}{c}(\beta y_0\Omega_u(u_0)p + (\beta\Omega(u_0)+\lambda)r). \tag{18}$$

The flow reduced to this set is

$$\begin{aligned}
p' &= q,\\
q' &= -cq + (\lambda - y_0\Omega_u(u_0))p - \Omega(u_0)r,\\
r' &= \frac{1}{c}(\beta y_f\Omega_u(u_0)p + (\beta\Omega(u_0)+\lambda)r),
\end{aligned} \tag{19}$$

together with the equation for τ: $\tau' = \kappa(1-\tau^2)$.

The equation $\dot{P} = A(\tau,\varepsilon,\lambda)P$ induces a flow on the space $\Lambda^k(\mathbb{C}^4)$

$$\dot{Y} = A^{(k)}(\tau,\varepsilon,\lambda)(Y). \tag{20}$$

Our goal now is to construct an invariant manifold for (20) which depends on ε continuously. To do so, we choose to work not with (20) directly, but with its conjugate. Indeed, for $Y \in \Lambda^k(\mathbb{C}^n)$ one can consider its Hodge star $*Y \in \Lambda^{n-k}(\mathbb{C}^n)$ (see [10, Sect.1.7] or [28, Ch.V]). The following statement [9, Prop.2] holds: If Y satisfies (20), then $(*Y)$ satisfies the conjugate equation

$$*\dot{Y} = [\overline{\mathrm{Trace}(A(\tau,\varepsilon,\lambda))}I_{n-k} - (A^{(n-k)}(\tau,\varepsilon,\lambda))^*](*Y),$$

where I_{n-k} is the identity on $\Lambda^{n-k}(\mathbb{C}^n)$ and $(A^{(n-k)}(\tau,\varepsilon,\lambda))^*$ is defined as in [9].

Based on the dimension of (18), we consider the case $k = 3$ and $n = 4$: $*Y \in \Lambda^1(\mathbb{C}^4)$ satisfies $\dot{Y} = A^{(3)}(\tau,\varepsilon,\lambda)(Y)$, $Y \in \Lambda^3(\mathbb{C}^4)$ if and only if

$$*\dot{Y} = [\overline{\mathrm{Trace}(A(\tau,\varepsilon,\lambda))}I_1 - (A^{(1)}(\tau,\varepsilon,\lambda))^*](*Y),\ *Y \in \Lambda^1(\mathbb{C}^4),$$

where $\mathrm{Trace}(A(\tau,\varepsilon,\lambda)) = -c(1+\varepsilon)$. This equation gives a system on $\mathbb{C}^4\times[-1,1]\times\mathbb{C}$

$$\begin{aligned}
*\dot{y}_1 &= -c(1+\varepsilon)(*y_1) - \varepsilon(\overline{\lambda} - y_f\Omega_u(u_f))(*y_2) - \beta y_f\Omega_u(u_f)(*y_4),\\
*\dot{y}_2 &= -\varepsilon(*y_1) - c(*y_2),\\
*\dot{y}_3 &= -\varepsilon\Omega(u_f)(*y_2) - (c(1+\varepsilon)+\varepsilon)(*y_3) + \beta(\overline{\lambda} - \Omega(u_f))(*y_4),\\
*\dot{y}_4 &= -\varepsilon c(*y_4),\\
\dot{\tau} &= \varepsilon\kappa(1-\tau^2),\\
\dot{\overline{\lambda}} &= 0,
\end{aligned}$$

or, on the slow scale,

$$\begin{aligned}
\varepsilon(*y_1)' &= -c(1+\varepsilon)(*y_1) - \varepsilon(\overline{\lambda} - y_f\Omega_u(u_f))(*y_2) - \beta y_f\Omega_u(u_f)(*y_4),\\
\varepsilon(*y_2)' &= -\varepsilon(*y_1) - c(*y_2),\\
\varepsilon(*y_3)' &= -\varepsilon\Omega(u_f)(*y_2) - (c(1+\varepsilon)+\varepsilon)(*y_3) + \beta(\overline{\lambda} - \Omega(u_f))(*y_4),\\
\varepsilon(*y_4)' &= -\varepsilon c(*y_4),\\
\tau' &= \kappa(1-\tau^2),\\
\overline{\lambda}' &= 0.
\end{aligned}$$

When $\varepsilon = 0$, the fast equation

$$(*\dot{Y}) = [\overline{\mathrm{Trace}(A(\tau,0,\lambda))}I_1 - (A^{(1)}(\tau,0,\lambda))^*](*Y) \tag{21}$$

together with the equations for τ and λ can be written as:

$$\begin{aligned}
*\dot{y}_1 &= -c(*y_1) - \beta y_f\Omega_u(u_f)(*y_4),\\
*\dot{y}_2 &= -c(*y_2),\\
*\dot{y}_3 &= -c(*y_3) + \beta(\overline{\lambda} - \Omega(u_f))(*y_4),\\
*\dot{y}_4 &= 0,\\
\dot{\tau} &= 0,\\
\dot{\overline{\lambda}} &= 0,
\end{aligned}\tag{22}$$

or in the fast coordinates:

$$\begin{aligned}
0 &= -c(*y_1) - \beta y_f\Omega_u(u_f)(*y_4),\\
0 &= -c(*y_2),\\
0 &= -c(*y_3) + \beta(\overline{\lambda} - \Omega(u_f))(*y_4),\\
*y_4' &= -c(*y_4),\\
\tau' &= \kappa(1-\tau^2),\\
\overline{\lambda}' &= 0.
\end{aligned}$$

There exists a 3-dimensional (parameterized by y_4, τ and $\overline{\lambda}$) manifold M of equilibria of (21)

$$*y_1 = -\frac{\beta}{c}y_f\Omega_u(u_f)(*y_4), \quad *y_2 = 0, \quad *y_3 = \frac{\beta}{c}(\overline{\lambda} - \Omega(u_f))(*y_4). \tag{23}$$

For every fixed $\tau\in[-1,1]$ and $\lambda\in\mathbb{C}$, this manifold is a point in the space of the Grassmannian manifold $G^{1,4}$ which is the projective space CP^3, and is compact.

A local coordinate chart for CP^3 may be defined by the scaling: $*\tilde{y}_1 = *y_1/*y_4$, $*\tilde{y}_2 = *y_2/*y_4$, $*\tilde{y}_3 = *y_3/*y_4$. In these local coordinates, (23) is given by

$$*\tilde{y}_1 = -\frac{\beta}{c} y_f \Omega_u(u_f), \quad *\tilde{y}_2 = 0, \quad *\tilde{y}_3 = \frac{\beta}{c}(\overline{\lambda} - \Omega(u_f)).$$

The system (22) in these local coordinates reads

$$\begin{pmatrix} *\dot{\tilde{y}}_1 \\ *\dot{\tilde{y}}_2 \\ *\dot{\tilde{y}}_3 \end{pmatrix} = -c \begin{pmatrix} *\tilde{y}_1 \\ *\tilde{y}_2 \\ *\tilde{y}_3 \end{pmatrix} + \begin{pmatrix} -\beta y_f \Omega(u_f) \\ 0 \\ \beta(\overline{\lambda} - \Omega(u_f)) \end{pmatrix}, \quad \dot{\tau} = 0, \quad \dot{\overline{\lambda}} = 0. \tag{24}$$

We consider the linearization of (24) about (23), for any fixed $\tau \in [-1, 1]$, $\overline{\lambda} \in \mathbb{C}$, on $CP^3 \times [-1, 1] \times \mathbb{C}$

$$\begin{pmatrix} *\dot{\tilde{y}}_1 \\ *\dot{\tilde{y}}_2 \\ *\dot{\tilde{y}}_3 \end{pmatrix} = -c \begin{pmatrix} *\tilde{y}_1 \\ *\tilde{y}_2 \\ *\tilde{y}_3 \end{pmatrix}, \quad \dot{\tau} = 0, \quad \dot{\overline{\lambda}} = 0.$$

There are two zero eigenvalues which correspond to the dimension of the manifold of equilibria (23) parameterized by τ and $\overline{\lambda}$. There is also one negative eigenvalue, $-c$. It indicates that (23), for each fixed τ and λ, as a point in $CP^3 \times [-1, 1] \times \mathbb{C}$, is an attractor. Therefore (23) is an invariant and normally hyperbolic manifold for (24). By Fenichel's invariant manifold theory it persists under small perturbations. For our case it means the following. For $\varepsilon > 0$ but sufficiently small, the equation

$$(*\dot{Y}) = [\overline{\mathrm{Trace}(A(\tau, \varepsilon, \lambda))} I_1 - (A^{(1)}(\tau, \varepsilon, \lambda)^*](*Y), \quad \dot{\tau} = \varepsilon\kappa(1 - \tau^2), \quad \dot{\overline{\lambda}} = 0 \tag{25}$$

induces a flow on $CP^3 \times [-1, 1] \times \mathbb{C}$ which is a perturbation of (22) of order ε. Recall that the front (u_f, y_f) for this equation denotes the perturbed front. For the flow on $CP^3 \times [-1, 1] \times \mathbb{C}$, by Fenichel's theorem, there exists a manifold (a curve in $CP^3 \times [-1, 1] \times \mathbb{C}$ or $G^{1,4} \times [-1, 1] \times \mathbb{C}$) which is a perturbation of order ε of (23). It is invariant, normally hyperbolic, and attracting on the fast scale. We can decompactify it by unfolding in the y_4-direction.

Unfolding this ε-dependent manifold to a manifold $*M_\varepsilon$ in $\Lambda^1(\mathbb{C}^4)$ we obtain an invariant manifold for (25) which is also attracting on the fast scale. That means that, for small enough $\varepsilon > 0$, the linearization of (25) about the perturbed manifold has only eigenvalues with negative real part. We apply the Hodge star operator to the points in $*M_\varepsilon$ and obtain a manifold M_ε in $\Lambda^3(\mathbb{C}^4) \times [-1, 1] \times \mathbb{C}$. Since $\Lambda^1(\mathbb{C}^4)$ is isomorphic to $\Lambda^3(\mathbb{C}^4)$, the manifold M_ε is an invariant, attracting manifold for

$$\dot{Y} = A^{(3)}(\tau, \varepsilon, \lambda)(Y), \quad Y \in \Lambda^3(\mathbb{C}^4), \tag{26}$$

in $\Lambda^3(\mathbb{C}^4) \times [-1, 1] \times \mathbb{C}$.

Thus an invariant manifold for (26) exists. It depends on ε continuously and, moreover, it is attracting. A reduced system can be obtained by restricting (26) to M_ε. More precisely, the equations are a perturbation of order ε of (19), projectivized and with the equation for τ appended.

At this point the construction [1] of the Stability Index Bundle is applicable. In our case, we call the bundle slow, since, as it is shown above, it is constructed as a perturbation of the bundle corresponding to $\varepsilon = 0$ case.

Assume we have a bounded simply connected domain $\mathcal{K}$ in $\mathbb{C}$ such that its boundary, contour K, does not contain any of the eigenvalues of (6). For any $\lambda \in K$, the fibers of the bundle are defined by means of the global unstable manifold of the point $(p, q, r, \tau) = (0, 0, 0, -1)$.

The standard capping-off procedure [1] provides fibers at $\overline{\mathcal{K}} \times \{\tau = \pm 1\}$, where $\overline{\mathcal{K}}$ is $\mathcal{K}$ with its boundary K. There exists a small $\gamma > 0$ such that each of the limiting systems

$$\dot{Y} = A^{(3)}(\pm 1, \varepsilon, \lambda)(Y), \quad Y \in \Lambda^3(\mathbb{C}^4), \quad \tau = \pm 1,$$

with $\operatorname{Re} \lambda > -\gamma$, has exactly one eigenvalue with positive real part, which is given by the sum of 2κ (from the flow in τ direction) and two of the largest eigenvalues. Thus it has a unique unstable eigenvector $\eta_\varepsilon(\pm 1, \varepsilon, \lambda, \xi)$ at $\tau = \pm 1$ which provides the caps at $\overline{\mathcal{K}} \times \{\tau = \pm 1\}$.

We summarize the result.

Lemma 3.1. *The augmented unstable bundle coincides with the slow bundle.*

Proposition 1 is a direct consequence of Lemma 3.1.

4. **Index Contour.** To make a conclusion on existence of unstable eigenvalues based on the properties of the Stability Index Bundle, we want to construct the base of the bundle using a contour that encloses all of the unstable eigenvalues, but does not contain any of the zeroes of the extended Evans function. We also want this contour to be independent of ε. We prove the existence of such contour in Sect. 4.2. The proof is based on two facts: all of the zeroes of the extended Evans function are isolated; the unstable discrete spectrum belongs to a bounded region. The latter and the dependence of the boundary of this region on parameter β is described in Sect. 4.1. We start this section by recalling properties of the fronts that we use to study the location of the unstable discrete spectrum.

Lemma 4.1. *The traveling wave (u_f, y_f) with $c > 0$ has the following properties: $y_f(\xi) > 0$ and $u_f(\xi) > 0$ for any ξ, and the wave is monotone in the sense that $y_f'(\xi) > 0$, $u_f'(\xi) < 0$ for any ξ.*

These properties of the front have been proved in [6] for $1 > \varepsilon > 0$ (with ignition temperature) and in [21] (without ignition temperature which corresponds to our case) and in [20] for $\varepsilon = 0$. Monotonicity of the front for $\varepsilon > 0$ case has been also proved in [25] under the assumption $y_\varepsilon > 0$ and $u_\varepsilon > 0$.

4.1. **Unstable point spectrum.** We want to identify a region where the unstable point spectrum, i.e., isolated eigenvalues of finite multiplicity from the open right half-plane, might be located. For the case of $\varepsilon = 0$, the estimate on $|\lambda|$, $\operatorname{Re} \lambda \geq 0$, for which the eigenvalue problem

$$\lambda p = p_{\xi\xi} + c_0 p_\xi + \Omega(u_0) r + y_0 \Omega_u(u_0) p, \tag{27}$$

$$\lambda r = c_0 \tilde{r}_\xi - \beta \Omega(u_0) r - \beta y_0 \Omega_u(u_0) p \tag{28}$$

has a non-trivial solution from L^2, has been obtained in [26, Sect.3]. More precisely, it has been shown that for such λ

$$|\lambda| \leq \frac{c_0^2}{4} + \max\{y_0 \Omega_u(u_0)\} + \left(\int \Omega(u_0)^2 |h|^2 \right)^{1/2}, \tag{29}$$

$$\operatorname{Re} \lambda \leq \max\{y_0 \Omega_u(u_0)\} + \left(\int \Omega(u_0)^2 |h|^2 \right)^{1/2}, \tag{30}$$

where

$$h(\xi) = \exp\left[-\frac{\beta}{c_0}\int_z^\infty \Omega(u_0)\mathrm{d}s\right] \times \left(\int_\xi^\infty |\exp\left[-2\frac{\beta}{c_0}\int_z^\infty \Omega(u_0(s))\mathrm{d}s\right]| \,(y_0(z)\Omega_u(u_0(z)))^2\mathrm{d}\,z\right)^{\frac{1}{2}}.$$

In other words, there exists an, independent of λ, constant such that any λ from the closed right-half plane cannot exceed in absolute value a certain β-dependent constant. The proof of the estimate is based on the fact that a bound on $|r|$ in terms of $\|p\|_{\mathrm{L}^2}$ can be obtained by solving the first order equation (28) for r, and then this bound can be used in an energy estimate in (27) to obtain (29) and (30).

Introduction of a diffusion term $\varepsilon r_{\xi\xi}$ in the eigenvalue problem does not change the situation. There exists an, independent of ε, bound on the absolute value of the eigenvalues of the perturbed problem. To show that we will study a convenient scaling of the eigenvalue problem (6). We will write (6) as a first order system, but we will use a different spatial scale, $\tilde{\eta} = \frac{|\lambda|^{1/2}}{\sqrt{\varepsilon}}\xi$, and $\tilde{q} = \frac{1}{|\lambda|^{1/2}}q$, $\tilde{s} = \frac{1}{|\lambda|^{1/2}}s$,

$$\begin{aligned} p' &= \sqrt{\varepsilon}\tilde{q}, \\ \tilde{q}' &= \sqrt{\varepsilon}(-\frac{c}{|\lambda|^{1/2}}\tilde{q} - \frac{\Omega(u_f)}{|\lambda|}r - \frac{y_f\Omega_u(u_f)}{|\lambda|}p + \mathrm{e}^{\mathrm{i}\arg\lambda}\mathrm{p}), \\ r' &= \tilde{s}, \\ \tilde{s}' &= -\frac{c}{|\lambda|^{1/2}}\tilde{s} + \frac{\beta\Omega(u_f)}{|\lambda|}r + \frac{\beta y_f\Omega_u(u_f)}{|\lambda|}p + \mathrm{e}^{\mathrm{i}\arg\lambda}\mathrm{r}. \end{aligned} \tag{31}$$

Here, the derivative $'$ is taken with respect to $\tilde{\eta}$.

As $|\lambda| \to \infty$ the right hand side of (31) is approximated by

$$\begin{aligned} p' &= \sqrt{\varepsilon}\tilde{q}, \\ \tilde{q}' &= \sqrt{\varepsilon}\mathrm{e}^{\mathrm{i}\arg\lambda}\mathrm{p}, \\ r' &= \tilde{s}, \\ \tilde{s}' &= \mathrm{e}^{\mathrm{i}\arg\lambda}\mathrm{r}. \end{aligned} \tag{32}$$

For $\lambda \in \{|\arg\lambda| < \pi\}$ and $\varepsilon > 0$, the limiting system has two eigenvalues with positive real part $\mathrm{e}^{\mathrm{i}\arg\lambda/2}$, $\sqrt{\varepsilon}\mathrm{e}^{\mathrm{i}\arg\lambda/2}$ and two eigenvalues with negative real part $-\mathrm{e}^{\mathrm{i}\arg\lambda/2}$, $-\sqrt{\varepsilon}\mathrm{e}^{\mathrm{i}\arg\lambda/2}$. For brevity, we use matrix notation,

$$P' = B_{\lim}P, \quad P = (p, \tilde{q}, r, \tilde{s}), \quad B_{\lim} = \begin{pmatrix} 0 & \sqrt{\varepsilon} & 0 & 0 \\ \sqrt{\varepsilon}\mathrm{e}^{\mathrm{i}\arg\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\arg\lambda} & 0 \end{pmatrix} \tag{33}$$

The matrix A is diagonalizable, there exist coordinate frame $P = AX$ in which (33) has a form

$$X' = DX, \tag{34}$$

and (32) reads

$$X' = DX + AB_eA^{-1}X, \tag{35}$$

where B_e is the difference between the right hand side of (31) and $B_{\lim}$. Obviously, $B_e = O(|\lambda|^{-1/2})$ uniformly for ε that is not large. The eigenfuction matrix A has a nice structure, so it is easy to check that $AB_eA^{-1} = O(|\lambda|^{-1/2})$ uniformly as long as ε is not large.

Since the unstable eigenspace is two dimensional, we consider the linear derivation of (34) on $\Lambda^2(\mathbb{C})$,

$$Z' = D^{(2)} Z. \tag{36}$$

The eigenvalue of $D^{(2)}$ are sums of two eigenvalues of D. The eigenvalue of $D^{(2)}$ with the largest positive real part is equal to $\mu_+ = \mathrm{e}^{\mathrm{i}\arg\lambda/2}(1+\sqrt{\varepsilon})$. The corresponding unstable eigenspace of (33) is a point, call it e_1, in $\Lambda^2(\mathbb{C})$. Projectivizing in the direction of the unstable eigenspace yields the flow induces by (36) on $\Pi(\Lambda^2(\mathbb{C}))$. The flow is described in terms of projectivized coordinates $\tilde{z}_i = \frac{z_i}{z_1}$, where z_1 is the coordinate in the direction of e_1. In the new coordinates, the origin is an invariant point for the flow and is attracting, since its eigenvalues are equal to the difference between eigenvalues of $D^{(2)}$ and μ_+ and therefore all have negative real parts.

In the same way, (35) induces a flow on $\Pi(\Lambda^2(\mathbb{C}))$. If $|\lambda|$ is large enough then there is a neighborhood of the unstable eigenspace that is still attracting and positively invariant. The trajectory of the compactified flow in $\Lambda^2(\mathbb{C}) \times [-1, 1]$ that follows the unstable manifold stays in that neighborhood and, therefore, must approach the unstable manifold at $\tau = 1$. Since it will never reach the stable manifold at $\tau = 1$, λ cannot be an eigenvalue. This argument is continuous in ε and the radius of the confinement region for the eigenvalues of (31) is independent of ε.

A straightforward application of Proposition 1 implies that the bound on $|\lambda|$ for $\varepsilon > 0$ is a perturbation of order ε of the upper bound in (29) and (30). Obviously, since ε is small, say, $\varepsilon < 1$, a bound for $|\lambda|$ can be chosen independent of ε.

The estimate (30) above depends on β. We will make this dependence more transparent using the properties of (u_f, y_f) from Lemma 4.1.

The temporal eigenvalues are $\lambda \in \mathbb{C}$ such that system (6) has a nontrivial, localized at $\pm\infty$, solution (p, r). Here we consider system (6) with all $\varepsilon \geq 0$. The estimate on the real part of the eigenvalues that we obtain below is independent of ε. We multiply the first equation in (6) by $\bar{p}$ and the second one by $\bar{r}$ and integrate over the real axis. It is easy to see that $\int p_{\xi\xi}\bar{p} = -\int |p_\xi|^2$, $\mathrm{Re}\int p_\xi \bar{p} = 0$. Using Lemma 4.1, we then get

$$\begin{aligned}
&\mathrm{Re}\,\lambda \int (|p|^2 + |r|^2) \\
&\leq \int y_f \Omega_u(u_f)|p|^2 - \beta \int y_f \Omega_u(u_f)|r|^2 + \mathrm{Re}\int \Omega(u_f) r\bar{p} - \mathrm{Re}\int \beta y_f \Omega_u(u_f) p\bar{r} \\
&\leq \frac{1}{2}\int [(\beta+2) y_f \Omega_u(u_f) + \Omega(u_f)]|p|^2 + \frac{1}{2}\int [(1-2\beta)\Omega(u_f) + y_f \Omega_u(u_f)]|r|^2.
\end{aligned}$$

Taking into account that $0 \leq y_f \leq 1$, $0 \leq u_f \leq 1/\beta$, and the fact that $x^2 e^{-x}$ on the interval $(0, +\infty)$ achieves maximum at $x = 2$ (think of $\beta < x = 1/u < \infty$) we estimate further:

$$\mathrm{Re}\,\lambda \leq \begin{cases} 17e^{-2}, & \text{if } \beta < 2; \\ \frac{1}{2}(4\beta^2+1)e^{-\beta}, & \text{if } \beta \geq 2. \end{cases}$$

This simple estimate provides an interesting piece of information. According to the numerics [2], when the parameter β is increased a pair of complex eigenvalues crosses the imaginary axis and moves into the unstable half-plane of the complex plane. The estimate shows that as β is increased even further, the eigenvalues somehow are pushed back to the imaginary axis. We summarize the results as a lemma.

Lemma 4.2. *Assume* λ, $\operatorname{Re}\lambda \geq 0$ *be such that (6), with* $\varepsilon \geq 0$, *has a non-trivial solution* $(p, q) \in L^2 \times L^2$. *Then there exist positive, independent of* ε, *constants* C_1 *and* C_2 *such that* $|\lambda| < C_1$ *and* $\operatorname{Re}\lambda \leq C_2$. *Moreover, as a function of* β, $C_2 = C_2(\beta)$ *is strictly decreasing for* $\beta \geq 2$ *and, when* $\varepsilon = 0$, $\lim_{\beta\to\infty} C_2(\beta) = 0$.

Here, for the case $\varepsilon > 0$, we do not discuss $\lim_{\beta\to\infty} C_2(\beta)$ because the upper bound on the size of ε, for which the existence and uniqueness of the front has been proved, generally speaking, depends on β. The nature of this dependence has not yet been investigated.

4.2. **Construction of a contour.** We want to show that it is possible to construct a closed contour which would enclose all of the unstable eigenvalues λ of the linearization of the system about the front with $\operatorname{Re}\lambda \geq 0$ and does not go through any of eigenvalues of either (u_0, y_0) or $(u_\varepsilon, y_\varepsilon)$. We base the construction of such a contour on two facts. On the right of the imaginary axis we use Lemma 4.2 according to which there exists a constant C, independent of ε, such that $|\lambda| \leq C$ for $\operatorname{Re}\lambda \geq 0$. The situation on the left of the imaginary axis is not that simple. To describe the contour for $\operatorname{Re}\lambda < 0$ we use the Evans function.

For any contour not crossing any of the zeroes of the Evans function, the first Chern number of the Stability Index Bundle over that contour coincides with the number of the zeroes of the Evans function inside of the contour [1].

Over the regions covered with the essential spectrum, the relation between the first Chern number of the Stability Index Bundle and the number of zeroes of the extended Evans function is still valid, but zeroes of the extended Evans function are not necessarily eigenvalues.

On the other hand, eigenvalues λ (embedded in the essential spectrum or not) are zeroes of the extended Evans function. As zeroes of an analytic function they are isolated, therefore it is possible to draw a contour which does not go through any of them. Moreover, from the construction of the Stability Index Bundle in Sect. 3 we see that any zero of the extended Evans function, if $\varepsilon > 0$ is sufficiently small, is located in a small neighborhood of a zero of the extended Evans function for a problem with $\varepsilon = 0$. Therefore, if we assume that the Evans function does not have any non-trivial, purely imaginary zeroes when $\varepsilon = 0$, then there exists $\varepsilon_0 > 0$ such that there is a contour independent of ε that encloses the unstable eigenvalues of the eigenvalue problems with all $0 \leq \varepsilon \leq \varepsilon_0$, and, at the same time, does not enclose or go through any of the zeroes $\lambda \neq 0$, $\operatorname{Re}\lambda \leq 0$ of the Evans functions corresponding to $0 \leq \varepsilon < \varepsilon_0$.

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E-mail address: ghazarya@email.unc.edu
E-mail address: ckrtj@email.unc.edu