Central-Upwind Scheme for a Non-hydrostatic Saint-Venant System

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Abstract

We develop a second-order central-upwind scheme for the non-hydrostatic version of the Saint-Venant system recently proposed in [M.-O. Bristeau and J. Sainte-Marie, Discrete Contin. Dyn. Syst. Ser. B, 10 (2008), pp. 733–759]. The designed scheme is both well-balanced (capable of exactly preserving the “lake-at-rest” steady state) and positivity preserving. We then use the central-upwind scheme to study ability of the non-hydrostatic Saint-Venant system to model long-time propagation and on-shore arrival of the tsunami-type waves. We discover that for a certain range of the dispersive coefficients, both the shape and amplitude of the waves are preserved even when the computational grid is relatively coarse. We also demonstrate the importance of the dispersive terms in the description of on-shore arrival.

Key words: hyperbolic systems of balance laws, dispersive shallow water systems, Godunov-type central-upwind schemes.

1 Introduction

Tsunami waves are characterized by having a relatively low amplitude, large wavelength, and large characteristic wave speed, see, e.g., [8, 25, 30]. In fact, the amplitude of a tsunami wave can be so small that it may not even be noticed by a ship traveling through it in deep water. Because of their speed and wavelength, however, these waves contain a tremendous amount of energy. When the depth of the water decreases (in the beginning of the on-shore arrival stage of tsunami wave propagation), tsunamis undergo a process called wave shoaling, in which the wave slows down and the wavelength decreases. In order to conserve energy, it is transformed from kinetic to potential energy and the wave amplitude increases. This potential energy can then be released in disastrous fashion when the wave comes to shore. It is therefore very important to have accurate models and corresponding numerical methods for tsunami waves in order to mitigate any catastrophe that may result.

One model used for shallow water waves is the classical Saint-Venant system [11], which is a depth-averaged system that can be derived from the Navier-Stokes equations (see, e.g., [13]).

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The Saint-Venant system is a very good simplification for lakes, rivers, and coastal areas in which the typical time and space scales of interest are relatively short. Tsunami waves form in deep water and travel very long distances (thousands of kilometers) before coming to shore. Over long time, solutions of the Saint-Venant system break down, dissipate in an unphysical manner, shock waves develop, and the system fails to capture small, trailing waves that are seen in nature and laboratory experiments [28]. Thus, it is necessary to use a more sophisticated model in order to preserve the wave characteristics over long time simulations.

Non-hydrostatic models (the celebrated Green-Naghdi equation [15] and several others, see, e.g., [1, 3, 4] and references therein) work well for long-time propagation of tsunami-like waves because they allow the wave to travel for long distances without decaying in amplitude. In addition, since these systems are dispersive, they give rise to trailing waves that are observed to follow tsunamis in nature. However, it is necessary to achieve some balance between dispersion observed with a non-hydrostatic model and the dissipation seen in the classical Saint-Venant system.

The non-hydrostatic Saint-Venant system presented in [5, 6] is given by

\[
\begin{align*}
    h_t + (hu)_x &= 0, \\
    (hu)_t + M_t + \left( hu^2 + \frac{g}{2} h^2 \right)_x + N &= -ghB_x + p^a w_x - 4(\nu u_x)_x - \kappa(h, hu)u,
\end{align*}
\]  

where \( h \) is the water depth measured vertically from the bottom topography function \( B(x, t) \), \( hu \) is the horizontal momentum or discharge, \( u(x, t) \) is the vertically averaged velocity, \( p^a = p^a(x, t) \) is the atmospheric pressure function, \( w := h + B \) is the free surface, \( \nu \) is the viscosity coefficient, \( \kappa \) is the friction function, and \( M \) and \( N \) are defined as

\[
M(h, hu, B) = \left( -\frac{1}{3} h^3 u_x + \frac{1}{2} h^2 B_x u \right)_x + B_x \left( -\frac{1}{2} h^2 u_x + B_x hu \right),
\]

and

\[
N(h, hu, B) = \left( (h^2)_t(h u_x - B_x u) \right)_x + 2B_x h_t(h u_x - B_x u) - B_xt \left( -\frac{1}{2} h^2 u_x + B_x hu \right).
\]  

(1.2)

Here, \( M \) and \( N \) are terms that arise when deriving this system from the Euler equations including non-hydrostatic pressure terms [6].

One of the goals of the current work is to numerically study the effects of the dispersion terms present in the non-hydrostatic model (1.1). To this end, we introduce the new scaling parameters \( \alpha_M \) and \( \alpha_N \) as coefficients to \( M \) and \( N \) in (1.1). For the purpose of this work we will neglect fluid viscosity and friction by setting \( \nu \) and \( \kappa(h, hu) \) to be identically zero. In addition, we follow the approach in [17, 21] and rewrite our system in terms of the equilibrium variables \( w := h + B \) and \( q := hu \):

\[
\begin{align*}
    w_t + q_x &= 0, \\
    q_t + \alpha_M M_t + \left( \frac{q^2}{w - B} + \frac{g}{2}(w - B)^2 \right)_x + \alpha_N N &= -g(w - B)B_x + p^a w_x.
\end{align*}
\]  

(1.3)

When \( \alpha_M = \alpha_N = p^a \equiv 0 \), this system reduces to the classical Saint-Venant system, and as we increase these parameters, the amount of dispersion in our model increases and the effects of the lack of the hydrostatic pressure assumption should be apparent.
To study the non-hydrostatic effects, we design a highly accurate and robust numerical method for (1.3). A good scheme should be well-balanced (it should exactly preserve “lake-at-rest” steady-state solutions), it should preserve positivity of $h$, and it should be able to properly handle discontinuous/nonsmooth solutions. The system (1.3) presents challenges in the approximation and treatment of the higher order mixed derivatives in the non-hydrostatic terms. These terms lead to the need to solve a coupled semi-discrete system of ODEs. A good scheme should treat these terms in a manner in which the resulting system may be solved efficiently. In this paper, we develop a central-upwind scheme for (1.3) which possesses all of these features. We then use our method to examine the effects of the non-hydrostatic pressure terms on the propagation of waves over long times and on their on-shore arrival.

Central-upwind schemes (first introduced in [24] and further developed in [18,20]) are Godunov-type finite volume methods. They belong to the class of Riemann-problem-solver-free central schemes and thus can be applied to a variety of hyperbolic systems of conservation laws as a “black-box” solver. When central-upwind schemes are applied to systems of balance laws, a special treatment of the source terms appearing in the system at hand must be developed. This was done for single- and two-layer shallow water models in [9,17,21–23]. To apply the central-upwind scheme to (1.3), one needs to specify the way the terms on the right-hand side (RHS) of (1.3) are discretized. This should be done in such a way that physically relevant steady-state solutions are exactly preserved and $h$ is guaranteed to be nonnegative.

The physically relevant steady-state solution for (1.3) is the “lake-at-rest” solution, corresponding to the water surface being perfectly flat and stationary:

$$w = h + B \equiv \text{const}, \quad hu \equiv 0. \quad (1.4)$$

Preserving this particular steady state would guarantee that no artificial surface waves are generated, and also ensure that small perturbations of the water surface will not lead to a “numerical storm”. This is achieved by using a special discretization of the geometric source term on the RHS of (1.3) which is presented in Section 2.1.3.

Preserving positivity of $h$ is essential since solutions containing negative $h$ would not only be unphysical, but will cause the numerical computations to fail. To ensure positivity of $h$, we follow the idea from [21]. We first replace the bottom topography with its continuous piecewise linear approximation and then adjust the piecewise linear reconstruction of the water heights, ensuring that through each computational cell the depth of each layer is nonnegative. This is presented in Section 2.1.1.

With the numerical method in place, we examine the effect of the non-hydrostatic pressure terms in Section 3, where we try to strike a balance between dissipation and dispersion inherent in the system.

## 2 Numerical Method

### 2.1 Central-Upwind Scheme

We develop a new well-balanced positivity preserving scheme for (1.3), which is based on the semi-discrete central-upwind scheme from [20] (see also [21,23]). For simplicity, we introduce a uniform grid $x_j = j\Delta x$ where $\Delta x$ is a small spatial scale, and denote the computational cells centered at $x_j$ by $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. 
We rewrite the system (1.3) in the following form:

\[
U_t + \mathcal{M}(U, B)_t + F(U, B)_x + N(U, B) = S(U, B), \quad U := (w, q)^T \tag{2.1}
\]

where

\[
F(U, B) = \left(q, \frac{q^2}{w-B} + \frac{g}{2}(w-B)^2\right)^T, \quad S(U, B) = (0, -g(w-B)B_x + p^\circ w_x)^T,
\]

\[
\mathcal{M}(U, B) = (0, \alpha_M \mathcal{M}(U, B))^T, \quad N(U, B) = (0, \alpha_N \mathcal{N}(U, B))^T.
\]

Using these notations, a semi-discrete finite-volume scheme for (2.1) takes the form of the following system of time-dependent ODEs:

\[
\frac{d}{dt}(\overline{U}_j(t) + \overline{M}_j(t)) = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \overline{S}_j(t) - \overline{N}_j(t), \tag{2.2}
\]

where \(\overline{U}_j(t)\) are approximations of the cell averages of the solution over the corresponding cells,

\[
\overline{U}_j(t) \approx \frac{1}{\Delta x} \int I_j U(x, t) dx,
\]

\(H_{j+\frac{1}{2}}(t)\) are numerical fluxes, and \(\overline{S}_j(t)\) is the discretization of the cell averages of the geometric source term:

\[
\overline{S}_j(t) \approx \frac{1}{\Delta x} \int I_j S(U(x, t), B(x)) dx.
\]

We use the central-upwind numerical fluxes \(H_{j+\frac{1}{2}}\) proposed in [21] (see also [18, 20, 23]):

\[
H_{j+\frac{1}{2}}(t) = \frac{a^+_j - a^-_j}{a^+_j - a^-_j} \left[ U^+_j - U^-_j \right], \tag{2.3}
\]

Here, the values \(U^\pm_j\) are the right/left point values at \(x = x_j + \frac{1}{2}\) of the conservative piecewise linear reconstruction \(\tilde{U}\),

\[
\tilde{U}(x) := \overline{U}_j + (U_x)_j (x - x_j), \quad x_j - \frac{1}{2} < x < x_j + \frac{1}{2}, \tag{2.4}
\]

which is used to approximate \(U\) at time \(t\), that is,

\[
U^\pm_{j + \frac{1}{2}} := \tilde{U}(x_j + \frac{1}{2} \pm 0) = \overline{U}_{j + \frac{1}{2} \pm 1} \pm \frac{\Delta x}{2} (U_x)_{j + \frac{1}{2} \pm 1}. \tag{2.5}
\]

The numerical derivatives \((U_x)_j\) are at least first-order accurate component-wise approximations of \(U_x(x, t)\), computed using a nonlinear limiter needed to ensure the non-oscillatory nature of the reconstruction (2.4). The right- and left-sided local speeds \(a^\pm_{j + \frac{1}{2}}\) in (2.3) are obtained from the smallest and largest eigenvalues of the Jacobian \(\frac{\partial F}{\partial U}\) (see Section 2.1.1 for details). Notice that the terms \((U^\pm_{j + \frac{1}{2}}, U_j, a^\pm_{j + \frac{1}{2}}, \tilde{U}(x)\) and \((U_x)_j\) all depend on \(t\), but we suppress this dependence for simplicity.

We also follow the work of [21, 23] and replace \(B(x)\) in (2.3) with its continuous piecewise linear approximation by defining

\[
B_{j + \frac{1}{2}} := B(x_{j + \frac{1}{2}}) \quad \text{and} \quad B_j := \frac{1}{2} (B_{j + \frac{1}{2}} + B_{j - \frac{1}{2}}). \tag{2.6}
\]

This will help to ensure the positivity preserving nature of the proposed scheme, as we show below.
2.1.1 Positivity-Preserving Reconstruction

The use of a piecewise linear reconstruction (2.4) requires the computation of slopes \((U_x)_j\) to obtain the right/left point values defined in (2.5). It is well-known that in order to ensure the non-oscillatory nature of the reconstruction, the use of a nonlinear limiter is required. We choose to use the generalized minmod limiter:

\[
(U_x)_j = \minmod \left( \frac{\theta (U_j - U_{j-1})}{\Delta x}, \frac{U_{j+1} - U_{j-1}}{2\Delta x}, \frac{\theta (U_{j+1} - U_j)}{\Delta x} \right), \quad \theta \in [1, 2],
\]

where the minmod function defined as

\[
\minmod(z_1, z_2, \ldots) := \begin{cases}
\min_j \{z_j\}, & \text{if } z_j > 0 \text{ } \forall j, \\
\max_j \{z_j\}, & \text{if } z_j < 0 \text{ } \forall j, \\
0, & \text{otherwise},
\end{cases}
\]

is applied in a componentwise manner. The parameter \(\theta\) can be used to control the amount of numerical viscosity present in the resulting scheme (see, e.g., [26, 29, 33] for more details concerning the generalized minmod and other nonlinear limiters).

Even when all of the cell averages \(h_j\) are nonnegative, the reconstructed right/left point values at the cell interface \(h_{j+\frac{1}{2}}^\pm\) may be negative. To guarantee positivity of \(h\) throughout the entire computational domain, we follow the procedure from [21] and amend the reconstruction (2.4), (2.5), (2.7) in the following conservative way:

\[
\begin{align*}
\text{if } w_{j+\frac{1}{2}}^- < B_{j+\frac{1}{2}}, & \text{ then take } (w_x)_j := -\frac{w_j}{\Delta x/2} \implies w_{j+\frac{1}{2}}^- = B_{j+\frac{1}{2}}, \quad w_{j-\frac{1}{2}}^+ = 2w_j; \\
\text{if } w_{j-\frac{1}{2}}^+ < B_{j-\frac{1}{2}}, & \text{ then take } (w_x)_j := \frac{w_j}{\Delta x/2} \implies w_{j+\frac{1}{2}}^- = 2w_j, \quad w_{j-\frac{1}{2}}^+ = B_{j-\frac{1}{2}}. 
\end{align*}
\]

It is necessary to compute the nonconservative quantity \(u = q/h\) for the computation of numerical fluxes and local propagation speeds. We follow the desingularization procedure outlined in [21, 23] to avoid possible division by small values of \(h\):

\[
u := \sqrt{\frac{2(w - B) \cdot q}{\sqrt{(w - B)^4 + \max(4(w - B)^4, \varepsilon)}}},
\]

where \(\varepsilon\) is a small desingularization parameter (in our numerical experiments, we have taken \(\varepsilon = \min((\Delta x)^2, 10^{-4})\)). Notice that this procedure will only affect the velocity computations when \(h^4 < \varepsilon\). It is also important to recalculate the values of \(q\) at the points where the velocity was desingularized by setting

\[q := h \cdot \nu.\]

Since the flux term \(F\) in (2.1) is equivalent to that of the classical Saint-Venant system, the local propagation speeds \(a_{j+\frac{1}{2}}^\pm\) are computed the same way using the eigenvalues of \(\frac{\partial F}{\partial U}\):

\[
a_{j+\frac{1}{2}}^+ := \max \left\{ u_{j+\frac{1}{2}}^+, \sqrt{gh_{j+\frac{1}{2}}^+}, u_{j+\frac{1}{2}}^- + \sqrt{gh_{j+\frac{1}{2}}^-}, 0 \right\},
\]

\[
a_{j+\frac{1}{2}}^- := \min \left\{ u_{j+\frac{1}{2}}^+, \sqrt{gh_{j+\frac{1}{2}}^+}, u_{j+\frac{1}{2}}^- - \sqrt{gh_{j+\frac{1}{2}}^-}, 0 \right\}.
\]

Remark 2.1 Proof of the positivity preserving property of this reconstruction is available in [17, 21].
2.1.2 Discretization of the Non-hydrostatic Pressure Terms

We follow [5] and discretize the terms of \( M \) at \( x_j \) in the following ways:

\[
\left( \frac{1}{3} h^3 u_x \right)_x (x_j) \approx \frac{1}{3 \Delta x} \left[ \frac{u_{j+1} - u_j}{\Delta x} \left( h_{j+\frac{1}{2}} \right)^3 - \frac{u_j - u_{j-1}}{\Delta x} \left( h_{j-\frac{1}{2}} \right)^3 \right]
\]

\[
= \frac{1}{3(\Delta x)^2} \left[ \left( h_{j+\frac{1}{2}} \right)^3 \bar{q}_{j+1} - \frac{1}{h_j} \left( h_{j+\frac{1}{2}} \right)^3 + \left( h_{j-\frac{1}{2}} \right)^3 \bar{q}_j + \frac{1}{h_{j-1}} \left( h_{j-\frac{1}{2}} \right)^3 \bar{q}_{j-1} \right], \quad (2.10)
\]

\[
\left( \frac{1}{2} h^2 B_z u \right)_x (x_j) = \left( \frac{1}{2} h B_z q \right)_x (x_j) \approx \frac{1}{2 \Delta x} \left[ h_{j+\frac{1}{2}} (B_z)_j + \frac{1}{2} q_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} (B_z)_j - \frac{1}{2} q_{j-\frac{1}{2}} \right]
\]

\[
= \frac{1}{4 \Delta x} \left[ h_{j+\frac{1}{2}} (B_z)_j + \frac{1}{2} \bar{q}_{j+1} + \left( h_{j+\frac{1}{2}} (B_z)_j + \frac{1}{2} - h_{j-\frac{1}{2}} (B_z)_j - \frac{1}{2} \right) \bar{q}_j - h_{j-\frac{1}{2}} (B_z)_j - \frac{1}{2} \bar{q}_{j-1} \right], \quad (2.11)
\]

\[
\left( \frac{1}{2} B_z h^2 u_x \right)(x_j) \approx \frac{1}{2} (B_z)_j \bar{r}_j^2 (u_x)_j \approx \frac{1}{2} (B_z)_j \bar{r}_j \left[ \frac{1}{\bar{r}_j} (q_x)_j - \frac{(h_x)_j}{h_j} \bar{q}_j \right]
\]

\[
= \frac{1}{4 \Delta x} (B_z)_j \left[ \bar{r}_j \bar{q}_{j+1} - 2 \Delta x (h_x)_j \bar{q}_j - \bar{r}_j \bar{q}_{j-1} \right], \quad (2.12)
\]

\[
\begin{aligned}
(B^2_h u)(x_j) &\approx (B_x)_j^2 \bar{q}_j, \\
\end{aligned} \quad (2.13)
\]

where \( u_j := \bar{q}_j / \bar{r}_j \) and

\[
\begin{aligned}
u_{j+\frac{1}{2}} := \frac{1}{2} (u_{j+1} + u_j), &\quad h_{j+\frac{1}{2}} := \frac{1}{2} (\bar{h}_{j+1} + \bar{h}_j), \quad q_{j+\frac{1}{2}} := \frac{1}{2} (\bar{q}_{j+1} + \bar{q}_j), \\
(B_x)_j := \frac{B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}}}{\Delta x}, &\quad (B_z)_j := \frac{1}{2} ((B_z)_j + (B_z)_j), \quad (q_x)_j := \frac{\bar{q}_{j+1} - \bar{q}_{j-1}}{2 \Delta x}. \\
\end{aligned} \quad (2.14)
\]

**Remark 2.2** In equations (2.10)–(2.13), \( (h_x)_j \) are obtained using the limiter as it is described in Section 2.1.1, while \( (q_x)_j \) are calculated using the centered differences (see (2.14)). The latter is done to avoid the need to solve a nonlinear system of algebraic equations as we explain in Section 2.2.

**Remark 2.3** We would like point out that all of the terms in (2.10)–(2.13) will be taken at either \( t^n \) or \( t^{n+1} \) depending on a particular choice of the time evolution method for the numerical integration of the system (2.2). The manner in which these terms are combined and treated is presented in Section 2.2.

2.1.3 Well-Balanced Source Discretization

Our goal is to design a numerical scheme for (1.3) that exactly preserves the “lake-at-rest” steady-state solution (1.4). This is achieved by selecting a proper discretization of the geometric source term \( S^{(2)}_j \). Such a discretization was derived for the classical Saint-Venant system in [17], and since both \( M_j \) and \( N_j \) as defined in Section 2.1.2 vanish at this steady state, we use this discretization along with an additional atmospheric pressure term for our scheme:

\[
\frac{\bar{S}^{(2)}_j}{2} = -g \left( \frac{w^+_{j+\frac{1}{2}} - B_{j+\frac{1}{2}}}{\Delta x} + \frac{w^-_{j-\frac{1}{2}} - B_{j-\frac{1}{2}}}{\Delta x} \right) \cdot \frac{(B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}})}{\Delta x} + p^a \frac{w^+_{j+\frac{1}{2}} - w^-_{j-\frac{1}{2}}}{\Delta x}. 
\]
2.2 Time Evolution

We solve the semidiscrete system (2.2) by applying the third-order strong stability preserving Runge-Kutta (SSP-RK) method from [14], which can be written as a convex combination of three forward Euler steps. For the purpose of demonstration, we proceed by fully discretizing (2.2) according to the forward Euler method, and all results obtained from doing so also apply to the SSP-RK method used in all of our numerical experiments.

When fully discretized by the forward Euler method, the first component of (2.2) becomes

\[ \overline{w}_{j}^{n+1} = \overline{w}_{j}^{n} - \lambda \left( H_{j+\frac{1}{2}}^{(1)} - H_{j-\frac{1}{2}}^{(1)} \right) , \]

where \( \lambda = \Delta t / \Delta x \). Notice that (2.15) has no contribution from \( M, N \) or \( S \). Thus, we may advance the first component independently of the second to obtain the cell averages of \( w \) at the new time level, \( \{ \overline{w}^{n+1}_{j} \}_{j=1}^{N} \) (and thus \( \{ \overline{h}^{n+1}_{j} \}_{j=1}^{N} \) since \( \overline{h}^{n+1}_{j} := \overline{w}^{n+1}_{j} - B_{j} \), where \( B_{j} \) is given by (2.6)).

Recall that the second component of the non-hydrostatic pressure term \( N \) involves time derivatives of \( h = w - B \) (see equation (1.2)). Equipped with both \( \overline{h}^{n+1}_{j} \) and \( \overline{w}^{n}_{j} \), we approximate these time derivatives by

\[ (h_{j}^{n+\frac{1}{2}})^{2} = \frac{\overline{h}_{j}^{n+1} - \overline{h}_{j}^{n}}{\Delta t}, \quad \text{and} \quad (h_{j}^{n+\frac{1}{2}})^{2} = \frac{(\overline{w}_{j}^{n+1})^{2} - (\overline{w}_{j}^{n})^{2}}{\Delta t}, \]

and then include them in the discretization of \( N \) on the RHS of (2.2). Thus, the fully discretized version of the second component of (2.2) becomes

\[ \overline{q}_{j}^{n+1} + \alpha_{M} M_{j}^{n+1} = \overline{q}_{j}^{n} + \alpha_{M} M_{j}^{n} - \lambda \left( H_{j+\frac{1}{2}}^{(2)} - H_{j-\frac{1}{2}}^{(2)} \right) + \Delta t \overline{S}^{(2)}_{j} - \Delta t \alpha_{N} N_{j}^{n+\frac{1}{2}} , \]

where all terms on the RHS of (2.17) are taken at \( t = t^{n} \) except for \( N_{j} \), which is taken at time \( t = t^{n+\frac{1}{2}} \).

Combining (2.10)–(2.13) for the discretization of \( M \) at time level \( t^{n+1} \) and inserting this into the LHS of (2.17) leads to the tridiagonal system \( T = (\tau_{j}^{n+1})^{\ast} \), \( j = 1, \ldots, N, \ i = j - 1, j, j + 1 \) for \( \{ \overline{q}^{n+1}_{j} \} \):

\[ \overline{q}_{j}^{n+1} + \alpha_{M} M_{j}^{n+1} = \tau_{j-1,j}^{n+1} \overline{q}_{j-1}^{n+1} + \tau_{j,j}^{n+1} \overline{q}_{j}^{n+1} + \tau_{j+1,j}^{n+1} \overline{q}_{j+1}^{n+1} , \]

where

\[ \tau_{j-1,j}^{n+1} = \alpha_{M} \frac{\overline{h}_{j-1}^{n+1} (B_{x})_{j} - \overline{h}_{j}^{n+1} (B_{x})_{j-\frac{1}{2}}}{4 \Delta x} - \frac{(\overline{h}_{j}^{n+1})^{3}}{3 \overline{h}_{j-1}^{n+1} (\Delta x)^{2}} , \]

\[ \tau_{j,j}^{n+1} = 1 + \alpha_{M} \frac{\overline{h}_{j+\frac{1}{2}}^{n+1} (B_{x})_{j+\frac{1}{2}} - \overline{h}_{j}^{n+1} (B_{x})_{j-\frac{1}{2}}}{4 \Delta x} + \frac{(\overline{h}_{j+\frac{1}{2}}^{n+1})^{3} + (\overline{h}_{j-\frac{1}{2}}^{n+1})^{3}}{3 \overline{h}_{j}^{n+1} (\Delta x)^{2}} + \frac{(B_{x})_{j} h_{j}^{n+1}}{2} + \frac{(B_{x})_{j}^{2}}{4} , \]

\[ \tau_{j+1,j}^{n+1} = \alpha_{M} \frac{\overline{h}_{j}^{n+1} (B_{x})_{j+\frac{1}{2}} - \overline{h}_{j+1}^{n+1} (B_{x})_{j}}{4 \Delta x} - \frac{(\overline{h}_{j+1}^{n+1})^{3}}{3 \overline{h}_{j+1}^{n+1} (\Delta x)^{2}} . \]
Notice that the term $\overline{q}_j^n + \alpha_M M_j^n$ on the RHS of (2.17) is discretized in the same way, but at time level $t = t^n$.

Finally, we use (2.16) and (2.14) to discretize $N$ in the same way as it was done in [5]:

$$
\Delta t \alpha_N N_j^{n+\frac{1}{2}} = \frac{\alpha_N}{\Delta x} \left[ \left( (h_{j+\frac{1}{2}}^{n+1})^2 - (h_{j-\frac{1}{2}}^n)^2 \right) \left( h_{j+\frac{1}{2}}^n - \frac{u_j^n}{\Delta x} \right) - (B_x)_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^n \right] \\
- \left[ \left( (h_{j-\frac{1}{2}}^{n+1})^2 - (h_{j-\frac{3}{2}}^n)^2 \right) \left( h_{j-\frac{1}{2}}^n - \frac{u_j^n}{\Delta x} \right) - (B_x)_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^n \right] + 2(B_x)_j (\overline{h}^{n+1}_j - \overline{h}^n_j) \left[ (q_x)_j^n - \{(h_x)_j^n + (B_z)_j\} u_j^n \right].
$$

(2.19)

**Remark 2.4** The addition of the dispersive terms $M$ and $N$ does not affect the well-balanced property of the scheme because these terms vanish at the “lake-at-rest” steady state (1.4). The positivity-preserving property of the scheme is also unaffected because these terms do not appear in the first equation of (1.1).

### 2.2.1 Solution of the Tridiagonal System

We may write the LHS of (2.17) as described by (2.18) as $\mathcal{T} q^{n+1}$, where $q^{n+1}$ is the vector of the unknown cell averages $\{\overline{q}_j^{n+1}\}_{j=1}^N$. When using free boundary conditions, $\mathcal{T}$ will be strictly tridiagonal, and it is well-known that in this case, the linear algebraic system (2.17) can be efficiently solved using the LU decomposition (see, e.g., [10, 34] for details).

To study the long-time behavior of the developed central-upwind scheme, one may use relatively small computational domain subject to the periodic boundary conditions that allow us to reduce computational cost. When these boundary conditions are used, the $\mathcal{T}$ becomes circulant; it contains nonzero entries in the upper-right and lower-left corners. In this case, we may still take advantage of the banded structure of the matrix by implementing the Sherman-Morrison algorithm proposed in [32].

The matrix $\mathcal{T}$ defined in (2.18) takes the following form:

$$
\mathcal{T} := \begin{pmatrix}
\tau_{1,1}^{n+1} & \tau_{1,2}^{n+1} & \ldots & \tau_{1,N}^{n+1} \\
\tau_{2,1}^{n+1} & \tau_{2,2}^{n+1} & \ldots & \tau_{2,N}^{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{N,1}^{n+1} & \tau_{N,2}^{n+1} & \ldots & \tau_{N,N}^{n+1}
\end{pmatrix}.
$$

We then define the $N$-vectors

$$
r := \left( \frac{1}{2}, 0, \ldots, 0, 2\tau_{N,N}^{n+1} \right)^T \quad \text{and} \quad s := \left( \frac{1}{2}, 0, \ldots, 0, 2\tau_{1,1}^{n+1} \right)^T,
$$

and use them to rewrite $\mathcal{T} = \hat{\mathcal{T}} + rs^T$, where $\hat{\mathcal{T}}$ is strictly tridiagonal, and we may make use of the Sherman-Morrison formula [32]:

$$
\mathcal{T}^{-1} = \hat{\mathcal{T}}^{-1} - \frac{r s^T \hat{\mathcal{T}}^{-1} s}{1 + s^T \hat{\mathcal{T}}^{-1} r}.
$$

We thus solve our system directly by the following procedure (see [7, 27, 31] for details):
1. Solve $\hat{T}y = b^n$, where $b^n$ is the vector form of the RHS of (2.17);
2. Solve $\hat{T}z = r$;
3. Set $q^{n+1} = y + \frac{s^Ty}{1+s^Tz}z$.

We stress that this procedure only requires inverting the tridiagonal matrix $\hat{T}$, which can be efficiently done using the LU factorization.

3 Numerical Experiments

In the following experiments, we will examine the role that the non-hydrostatic pressure terms play in the long-time propagation of water waves. We will use the classical Saint-Venant system for comparison, which is simply (1.3) with $\alpha_M = \alpha_N = p^a \equiv 0$. In all of the experiments, we take $p^a \equiv 0$, take the minmod parameter $\theta = 1.3$, and consider the periodic boundary conditions.

3.1 Small Generated Wave Propagation

First, we consider a wave that was generated using a Savage-Hutter type model of submarine landslides and generated tsunami waves. This is a two-layer system in which the lower layer is considered to be a fluid-granular mixture that has a larger density than the upper layer, which is water. The lower layer slides down the slope of the solid bottom, and the through momentum exchange causes waves to form at the water surface. For more details of this system and associated numerical methods, see [12, 16, 19].

We run the generation simulation with the following bottom topography function:

$$B(x) = -\frac{3}{4} - \frac{1}{2} \left[ 1 - (2 - x) - \text{sgn}(2 - x) \left( 1 - (|2 - x|^c + 1)^{1/c} \right) \right], \quad (3.1)$$

which essentially describes two piecewise constant pieces for $x < 1$ and $x > 3$ connected by a linear piece where the corners are smoothed out, which is necessary because of limitations of the underlying Savage-Hutter type model (the smoothing parameter is taken to be $c = 10$). The water surface is initially flat and the lower layer sits on the sloped part of the bottom. Both layers are taken to be initially motionless. This is depicted in Figure 3.1.

Next, we take the wave at $t = 5$ to be the initial condition for the non-hydrostatic system (1.3). We adjust the computational domain by removing the portion where $B(x)$ was not flat, and extend it so that our new computational domain is $[10, 60]$. In this region, the lower layer was identically zero, so we now only need to consider a single layer non-hydrostatic model.

We first present time snapshots of the solutions for various values of the dispersion parameters $\alpha_M$ and $\alpha_N$. Figure 3.2 shows that as these parameters increase, the shape of the wave changes. The leading crest of the wave decreases while the trough becomes more pronounced. The secondary crest increases in amplitude and smaller trailing waves also form. As expected, the larger the non-hydrostatic parameters are, the more drastic the shape change is.

A long time evolution of the solution (for $t \in [0, 100]$) is shown in Figure 3.3. The parameters for this set of experiments have been chosen to illustrate the drastic difference between the numerical solutions of the hydrostatic ($\alpha_M = \alpha_N = 0$) and non-hydrostatic (with $\alpha_M = 0.05$ and $\alpha_N = 400$) Saint-Venant systems.
Figure 3.1: On the left: Bottom topography given by (3.1) and initial setting for the wave generation. There is a small ridge formed by the sediment layer that deforms and generates small surface waves. On the right: The right-moving wave at time $t = 5$. We take it to be the initial condition for the non-hydrostatic system (1.3).

To further emphasize the difference between these solutions, we let these two waves to approach the shore. To this end, we increase the computational domain to $[10, 110]$, replace the periodic boundary conditions with absorbing (free) boundary conditions and use the following piecewise linear continuous bottom topography function:

$$B(x) = \begin{cases} 
-1.75, & x < 75, \\
(1.75 - 10^{-8})(x - 76) - 10^{-8}, & 75 \leq x < 76, \\
10^{-8}(x - 76)/34 - 10^{-8}, & x \geq 76.
\end{cases}$$

To accurately capture the on-shore arrival of the waves, we have implemented a special technique from [2]. The obtained results are depicted in Figure 3.4. At time $t = 13$ the “non-hydrostatic” wave reaches its peak height, which is about 4 times larger than the peak on the “hydrostatic” wave reached much earlier (at about $t = 9.6$). By time $t = 15$ both the magnitude and the shape of the waves are of completely different nature.

### 3.2 Large-Scale Generated Wave Propagation

In this example, we examine a large scale tsunami wave that is again generated by the Savage-Hutter type model of submarine landslides using the numerical method described in [19], but on a much larger scale. The length scale is kilometers and the time scale is hours, so we take the corresponding gravity to be $g = 271008 \text{ km/h}^2$. In this example, a submarine landslide on the ocean floor creates surface waves traveling to the left and right. We choose to track the right-moving wave over long time propagation, which in this scenario we consider to around $t = 3$ since the time is now measured in hours.

Figure 3.5 shows that over long time propagation, the wave changes shape, forming trailing waves as it is observed in nature. These trailing waves are more pronounced for larger values of $\alpha_M$ and $\alpha_N$.

### 3.3 Non-flat Bottom Topography

In this section, we examine the impact of having a bottom topography function that is not constant. This reflects the fact that in the ocean the bottom topography is never flat; it always has some
Figure 3.2: Small time \((t = 5)\) snapshots of the numerical solutions of the non-hydrostatic system \((1.3)\) obtained on the computational domain \([10, 60]\) subject to the periodic boundary conditions and the initial data shown in Figure 3.1 (right). For each plot on the left, \(\alpha_N\) is fixed and solutions are shown for \(\alpha_M = 0.02, 0.04, 0.06, 0.08\) and 0.10. For each plot on the right \(\alpha_M\) is fixed and solutions are shown for \(\alpha_N = 0, 100\) and 200. Notice that while small changes in \(\alpha_M\) drastically change the shape of the wave, relatively large changes in \(\alpha_N\) have only a small impact on the solution.

hills and valleys. We take the same initial condition used in Section 3.2, but translate it to the left so that the leading edge of the wave begins near \(x = 0\) at time \(t = 0\). Again, we take \(g = 271008\), but use the following bottom topography function:

\[
B(x) = \begin{cases} 
-5, & x < 0, \\
-5 + \sum_{i=1}^{5} C_i \sin(\pi(x - S_i)/L_i), & x \geq 0,
\end{cases}
\]  

(3.2)

where the parameters \(C_i, S_i\) and \(L_i\) are given in Table 3.1. This bottom topography is depicted in Figure 3.6.

Figure 3.7 shows time snapshots of the numerical solutions of the non-hydrostatic system \((1.3)\) with \(\alpha_M = \alpha_N = 0.05, 0.1, 0.15\) and 0.2 along with the corresponding solutions of the classical Saint-Venant system \((\alpha_M = \alpha_N = 0)\). There are many small waves created behind the large wave as a result of the non-flat bottom topography, but the structure of the larger waves does not seem to be significantly affected.
Figure 3.3: Time snapshots of the numerical solutions of the hydrostatic ($\alpha_M = \alpha_N = 0$) and non-hydrostatic (with $\alpha_M = 0.05$ and $\alpha_N = 400$) Saint-Venant systems obtained on the computational domain [10, 60] subject to the periodic boundary conditions and the initial data shown in Figure 3.1 (right). The solution of the non-hydrostatic system (solid green line) maintains wave amplitude and forms trailing waves while drastically changing shape. The solution of the classical Saint-Venant system (dashed blue line) diffuses and decays over time.

Table 3.1: Parameters used in for the bottom topography function (3.2) shown in Figure 3.6.

<table>
<thead>
<tr>
<th>$i$</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
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<td>0.5</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>$S_i$</td>
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<td>0</td>
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<tr>
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<td>70</td>
<td>100</td>
<td>10</td>
<td>2500</td>
</tr>
</tbody>
</table>

3.3.1 On-shore Dynamics of the Large Wave

As in Section 3.1, to examine the ultimate effects of the nonhydrostatic pressure terms, we allow this wave to run up on shore (see [2] for details on methods required). We take the solutions at time $t = 2$ shown in Figure 3.7 for $\alpha_M = \alpha_N = 0$ and $\alpha_M = \alpha_N = 0.2$ as initial data on the domain [1000, 3000] along with following bottom topography function:

$$B(x) = \begin{cases} 
-5 + \sum_{i=1}^{5} C_i \sin(\pi(x - S_i)/L_i), & x < 2200, \\
\max\left(-4.86, -4.86 + 2.75 \exp\left[-300(1 - \frac{x}{2600})\right]\right), & 2200 < x < 2600, \\
10^{-10} - 2.11 \exp\left[-300(\frac{x}{2600} - 1)\right], & x > 2600,
\end{cases}$$

(3.3)

where the coefficients $C_i$, $S_i$, and $L_i$ are given in Table 3.1. This is simply a smooth curve that increases from $-4.86$ (where the nonflat topography leaves off) to a depth of zero.

At first glance (Figure 3.8), the dispersive ($\alpha_M = \alpha_N = 0.2$) and nondispersive ($\alpha_M = \alpha_N = 0$) waves do not seem to behave very differently near the shore. They each go through the shoaling
Figure 3.4: Time snapshots of the "non-hydrostatic" (solid green line) and "hydrostatic" (dashed blue line) waves coming to shore with the initial data taken to be the solutions at time $t = 100$ from Figure 3.3.

Figure 3.5: Time snapshots of the numerical solution with increasing values of $\alpha_M = \alpha_N$. The solutions are computed on a periodic domain with $\Delta x = 0.5$. In this example, the formation of trailing waves is the most pronounced difference between the classical Saint-Venant system and the non-hydrostatic one.
process where they slow down and increase in height, and eventually arrive on shore. If we look closer, however (Figure 3.9), we see that the trailing waves actually do impact how the wave comes to shore. The front of the non-hydrostatic solution is about 10 km ahead of the hydrostatic one. This suggests that the non-hydrostatic terms must be included in a tsunami model if one wants to accurately represent the ultimate outcome of the tsunami waves.

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References


Figure 3.7: Time snapshots of the evolution of a tsunami-like wave over non-flat bottom topography described by (3.2). Notice that the domain is shifting as the time increases in order to see the wave up close.


Figure 3.8: The resulting waves from Figure 3.7 for $\alpha_M = \alpha_N = 0$ and $\alpha_M = \alpha_N = 0.2$ are allowed to run up on shore. There does not seem to be a significant difference between the dispersive and nondispersive waves at this zoom level.
Figure 3.9: The same simulation as Figure 3.8, but zoomed in. At this zoom level, we can see that the dispersive \((\alpha_M = \alpha_N = 0.2)\) wave (solid line) is ahead of the nondispersive \((\alpha_M = \alpha_N = 0)\) one (dashed line) by about 10 km, which is quite a significant difference when the on-shore tsunami wave arrival is studied.
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