

MA 584 - Suggested Projects

due on December 16, 2008, 11:00am.

1. Search the Internet to find a fast Poisson solver. Implement, debug and test this algorithm. You should assume a Neumann or mixed boundary condition at least along one side of the domain. Chose two examples with known exact solutions to test and debug your code. One of the examples should be a quadratic function and at least one of the examples should have nonhomogeneous boundary conditions.
2. Solve the Poisson equation

$$\nabla \cdot (\beta \nabla u) - qu = f(x, y)$$

on an irregular domain, for example, bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $r = 0.5 + 0.1 \sin(5\theta)$ (in polar coordinates).

3. Derive and implement the ADI (alternating directional implicit) method for the heat equation with variable coefficients:

$$u_t = (a(x, y, t)u_x)_x + (a(x, y, t)u_y)_y + f(x, y, t),$$

assuming that $a(x, y, t) > 0$. You should assume a Neumann or mixed boundary condition at least along one side of the domain. Chose two examples with known exact solutions to test and debug your code. At least one of the examples should have nonhomogeneous boundary conditions.

4. Derive a generalization of Godunov's method for a system of conservation laws in 2-D:

$$u_t + f(u)_x + g(u)_y = 0.$$

Implement and test the algorithm. Find a stability limit of the two-dimensional version of Godunov's method you derived by considering the linear problem

$$u_t + u_x + u_y = 0$$

with data

$$U_j^0 = \begin{cases} 1, & i + j < 0, \\ 0, & i + j \geq 0. \end{cases}$$

5. Determine the von Neumann stability condition for the solution of the equation $u_t = u_{xx} + u_{yy} + f(x, y)$ by the three-term difference scheme

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n-1}}{2k} = \frac{U_{i,j+1}^n + U_{i,j-1}^n + U_{i+1,j}^n + U_{i-1,j}^n - 2U_{i,j}^n - U_{i,j}^{n-1}}{h^2} + f_{i,j}$$

on a uniform square mesh size $h = \Delta x = \Delta y$. Explain how this scheme might be used as an iterative method for the solution of the equations

$$\delta_x^2 U_{i,j} + \delta_y^2 U_{i,j} + h^2 f_{i,j} = 0,$$

where $\delta_x^2 v(x, t) := v(x + \Delta x, t) - 2v(x, t) + v(x - \Delta x, t)$. Show that taking $k = \frac{1}{4}h^2$ gives the same rate of convergence as Jacobi's method, but that the choice

$$k = \frac{h^2}{4 \sin(\pi h)}$$

gives significantly faster convergence. Run numerical experiments to illustrate the theoretical construction.