Fall 2008 A. Chertock

MA 584 - Homework #5

due on December 2, 2008

1. Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}). \tag{1}$$

for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with periodic boundary conditions.

   (a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

   (b) Suppose we want to fix the Courant number ak/h as $k,\ h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

   (c) Apply von Neumann stability analysis to the method (1). What is the amplification factor $g(\xi)$?

   (d) For what range of ak/h will the CFL condition be satisfied for this method (with periodic boundary conditions)?

   (e) Suppose we use the same method (1) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (1) can be applied at x_{m+1}). For what range of ak/h will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

   (f) Determine the modified equation on which the method (1) is second order accurate. Is this method predominantly dispersive or dissipative?

2. Derive the modified equation (10.45) (in the text book) for the Lax-Wendroff method.

3. The m-file `advection_LW_pbc.m` (available for the website) implements the Lax-Wendroff method for the advection equation on $0 \le x \le 1$ with periodic boundary conditions.

   (a) Observe how this behaves with $m+1 = 50,\ 100,\ 200$ grid points. Change the final time to `tfinal = 0.1` and use the m-files `error_table.m` and `error_loglog.m` to verify second order accuracy.

   (b) Modify the m-file to create a version `advection_up_pbc.m` implementing the upwind method and verify that this is first order accurate.

(c) Keep m fixed and observe what happens with `advection_up_pbc.m` if the time step k is reduced, e.g. try $k = 0.4h$, $k = 0.2h$, $k = 0.1h$. When a convergent method is applied to an ODE we expect better accuracy as the time step is reduced and we can view the upwind method as an ODE solver applied to an MOL system. However, you should observe decreased accuracy as $k \to 0$ with h fixed. Explain this apparent paradox.

Hint: What ODE system are we solving more accuracy? You might also consider the modified equation (10.44) in the text book.

4. Verify that the Richtmayer two-step Lax-Wendroff method,

$$U_{j+1/2}^{n+1/2} = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{k}{2h}\left[f(U_{j+1}^n) - f(U_j^n)\right]$$
$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[f(U_{j+1/2}^{n+1/2}) - f(U_{j-1/2}^{n+1/2})\right],$$

for a conservation law $u_t + f(u)_x = 0$ reduces to

$$U_j^{n+1} = U_j^n - \frac{k}{2h}a\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{k^2}{2h^2}a^2\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

in the constant coefficient linear case and is second order accurate on smooth solutions (to nonlinear problems) and conservative. Write the method in conservation form, determining the numerical flux function.

5. Consider the inviscid Burgers equation with the initial data given by

$$\eta(x) = \begin{cases} 0, & \text{if} \quad -1 \le x < -2/3, \\ 1, & \text{if} \quad -2/3 \le x < 0, \\ 0, & \text{if} \quad 0 \le x < 1, \end{cases}$$

and with periodic boundary conditions on $[-1, 1]$.

(a) Find the exact solution for $t \le 1$.

(b) Implement the upwind and Lax-Friedrichs schemes in conservative form as well as Godunov's scheme. Find the solutions using $h = 0.01$ and $h = 0.001$ and $k = 0.9h$. Compare against the exact solution at $t = 1$. What are your observations?

6. For a nonlinear conservation law, $u_t + f(u)_x = 0$, consider Godunov's finite-volume method:

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F_{j+1/2}^n - F_{j-1/2}^n\right].$$

Show that the Godunov flux function for a convex scalar conservation law ($f''(u) > 0$) is given by

$$F_{j+1/2}^n = \begin{cases} \min\limits_{U_j^n \le q \le U_{j+1}^n} f(q), & \text{if} \quad U_j^n \le U_{j+1}^n, \\ \min\limits_{U_{j+1}^n \le q \le U_j^n} f(q), & \text{if} \quad U_{j+1}^n \le U_j^n. \end{cases}$$

7. Consider the upwind method with flux

$$\mathcal{F}(v,w) = \begin{cases} f(v), & \text{if } (f(v)-f(w))/(v-w) \geq 0 \\ f(w), & \text{if } (f(v)-f(w))/(v-w) < 0. \end{cases}$$

Take $k/h = 1/2$ and apply to inviscid Burgers' equation with initial data

$$u_0(x) = \begin{cases} -1, & \text{if } x < 1 \\ 1, & \text{if } x > 1 \end{cases}$$

discretized using $U_j^0 = \bar{u}_j^0$ (cell averages). Justify the following statements:

(a) The sequence $U_l(x,t)$ for $k_l = 1/2l$ converges to the correct rarefaction wave solution as $l \to \infty$.

(b) The sequence $U_l(x,t)$ for $k_l = 1/(2l+1)$ converges to an entropy violating shock as $l \to \infty$.

(c) The sequence $U_l(x,t)$ for $k_l = 1/l$ does not converge as $l \to \infty$.