Fall 2008 A. Chertock

MA 584 - Homework #4

due on November 11, 2008

1. Verify the consistency and stability conditions of the θ-scheme,

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} = \frac{\kappa}{h^2}\left[\theta\nabla_h^2 U_j^{n+1} + (1-\theta)\delta_{xx}^2 U_j^n\right],$$

   for the heat equation $u_t = \kappa u_{xx}$.
   Here $\delta_{xx}^2 U_j^l = U_{j+1}^l - 2U_j^l + U_{j-1}^l,\ l = n, n+1$.

2. Show that the implicit scheme for the heat equation, $u_t = \kappa u_{xx}$, given by

$$\left(1-\frac{\kappa\Delta t}{2h^2}\delta_{xx}^2\right)\left(\frac{U_j^{n+1}-U_j^n}{\Delta t}\right) = \frac{\kappa}{h^2}\left(1-\frac{1}{12}\delta_{xx}^2\right)\delta_{xx}^2 U_j^n$$

   is accurate of order $\left((\Delta t)^2, h^4\right)$ and stable if $\dfrac{\kappa\Delta t}{h^2} \le \dfrac{3}{2}$.
   Again, $\delta_{xx}^2 U_j^l = U_{j+1}^l - 2U_j^l + U_{j-1}^l,\ l = n, n+1$.

3. Consider the advection-diffusion equation

$$u_t + u_x = \kappa u_{xx}, \quad \kappa > 0.$$

   Use the von Neumann analysis to derive the time step restriction for the following scheme

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} + \frac{U_{j+1}^n - U_{j-1}^n}{2h} = \kappa\,\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}.$$

4. **(codes for heat equation)**
   (a) The m-file `heat_CN.m` (available from the website) solves the heat equation $u_t = \kappa u_{xx}$ using the Crank-Nicolson method. Run this code, and by changing the number of grid points, confirm that it is second-order accurate. (Observe how the error at some fixed time such as $T = 1$ behaves as k and h go to zero with a fixed relation between k and h, such as $k = 4h$.)

   You might want to use the function `error_table.m` to print out this table and estimate the order of accuracy, and `error_loglog.m` to produce a log-log plot of the error vs. h. See `bvp_2.m` for an example of how these are used.

   (b) Modify `heat_CN.m` to produce a new m-file `heat_trbdf2.m` that implements the TR-BDF2 method, described on see page 175, in the text book, on the same problem. Test it to confirm that it is also second order accurate. Explain how you determined the proper boundary conditions in each stage of this Runge-Kutta method.

(c) Modify `heat_CN.m` to produce a new m-file `heat_FE.m` that implements the forward Euler explicit method on the same problem. Test it to confirm that it is $\mathcal{O}(h^2)$ accurate as $h \to 0$ provided when $k = 24h^2$ is used, which is within the stability limit for $\kappa = 0.02$. Note how many more time steps are required than with Crank-Nicolson or TR-BDF2, especially on finer grids.

(d) Test `heat_FE.m` with $k = 26h^2$, for which it should be unstable. Note that the instability does not become apparent until about time 1.6 for the parameter values $\kappa = 0.02$, $m = 39$, $\beta = 150$. Explain why the instability takes several hundred time steps to appear, and why it appears as a sawtooth oscillation.

Hint: What wave numbers ξ are growing exponentially for these parameter values? What is the initial magnitude of the most unstable eigenmode in the given initial data? The expression (16.52) for the Fourier transform of a Gaussian may be useful.

5. **(heat equation with discontinuous data)**

(a) Modify `heat_CN.m` to solve the heat equation for $-1 \le x \le 1$ with step function initial data

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x \ge 0. \end{cases} \tag{1}$$

With appropriate Dirichlet boundary conditions, the exact solution is

$$u(x,t) = \frac{1}{2}\,\mathrm{erfc}\left(x/\sqrt{4\kappa t}\right), \tag{2}$$

where erfc is the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\,dz.$$

i. Test this routine $m = 39$ and $k = 4h$. Note that there is an initial rapid transient decay of the high wave numbers that is not captured well with this size time step.

ii. How small do you need to take the time step to get reasonable results? For a suitably small time step, explain why you get much better results by using $m = 38$ than $m = 39$. What is the observed order of accuracy as $k \to 0$ when $k = \alpha h$ with α suitably small and m even?

(b) Modify `heat_trbdf2.m` to solve the heat equation for $-1 \le x \le 1$ with step function initial data as above. Test this routine using $k = 4h$ and estimate the order of accuracy as $k \to 0$ with m even. Why does the TR-BDF2 method work better than Crank-Nicolson?

6. Implement the ADI method for two dimensional heat equations

$$
\begin{aligned}
&u_t = u_{xx} + u_{yy} + f(x,y,t), \quad a < x < b, \quad c < y < d, \quad t > 0,\\
&u(x,y,0) = u_0(x,y),\\
&u_x(b,y,t) = g(y,t),
\end{aligned}
$$

and Dirichlet boundary condition at $x = a$, $y = c$, and $y = d$. You need to use the ghost point method to deal with the flux boundary condition. Choose a non-trivial exact solution and do the grid refinement analysis. Plot your solution and error using mesh and/or contour plots (you probably just need three plots) for a suitable mesh size.