Fall 2008 A. Chertock

MA 584 - Homework #3

due on October 21, 2008

1. Compute the leading term in the local truncation error of the following methods:

   (a) the trapezoidal method (5.22),

   (b) the 2-step BDF method (5.25).

2. Consider the Runge-Kutta method defined by the tableau below. Show that the method is third order accurate in two different ways: First by checking that the order conditions (5.35), (5.38), and (5.39) in the text book are satisfied, and then by applying one step of the method to $u' = \lambda u$ and verifying that the Taylor series expansion of $e^{k\lambda}$ is recovered to the expected order.

$$\begin{array}{c|cc} 0 & & \\ 1/3 & 1/3 & \\ 2/3 & 0 & 2/3 \\ \hline & 1/4 & 0 \end{array}$$

3. Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

   (a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$

   (b) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$

   (c) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

   Hint: The consistency condition for LMM is given by (5.48) in the text book. The definition of zero-stability is on page 147.

4. Show that the leap-frog method is unstable for any initial value problem of the form $u' = \lambda u$ with $\mathcal{R}e(\lambda) < 0$.

5. For a given ODE $u' = f(u)$, consider the so-called θ-method

$$U^{n+1} = U^n + k\left(\theta f(U^{n+1}) + (1-\theta)f(U^n)\right)$$

   for some value θ, $0 \le \theta \le 1$.

   (a) Which methods are obtained for the values (i) $\theta = 0$, (ii) $\theta = 1$, and (iii) $\theta = 1/2$?

   (b) Find a range of θ-values, i.e., an interval $[a, b]$ such that the method is A-stable for any $a \le \theta \le b$.

6. The problem

$$u' = u^2, \quad u(0) = 1,$$

has the exact solution

$$u(t) = \frac{1}{1-t}, \quad 0 \le t < 1.$$

This blows up at $t = 1$. Integrate the problem on the interval $[0, 1-\varepsilon]$ for a parameter $0 < \varepsilon \ll 1$ using (i) forward Euler and (ii) RK4 with a constant step size k. Try this for $k = \varepsilon = 0.1, 0.01, 0.001, 0.0001$. What do you observe? Is the difference between methods' behavior fundamental?

7. Consider problem described in Example 7.11 in the text book. Implement and test the midpoint, trapezoid, and Adams-Bashforth (AB2) methods (all of which are second order accurate) for each of the following case (and perhaps others of your choice) and comment on the behavior of each method.

   (a) $a = 100,\ b = 0$ (undamped),

   (b) $a = 100,\ b = 3$ (damped),

   (c) $a = 100,\ b = 10$ (more damped).

   The two-step Adams-Bashforth (AB2) method reads

$$U^{n+1} = U^n + \frac{k}{2}\left(3f(U^n, t_n) - f(U^{n-1}, t_{n-1})\right), \quad n \ge 1.$$

8. This problem can be solved by a modifying the m-files odesample.m and odesampletest.m available from the webpage.

   Consider the third order initial value problem

$$v'''(t) + v''(t) + 4v'(t) + 4v(t) = 4t^2 + 8t - 10,$$
$$v(0) = -3, \quad v'(0) = -2, \quad v''(0) = 2.$$

   (a) Verify that the function

$$v(t) = -\sin(2t) + t^2 - 3$$

   is a solution to this problem. How do you know it is the unique solution?

   (b) Rewrite this problem as a first order system of the form $\mathbf{u}'(t) = f(\mathbf{u}(t), t)$ where $\mathbf{u}(t) \in \mathbb{R}^3$. Make sure you also specify the initial condition $\mathbf{u}(0) = \eta$ as a 3-vector.

   (c) Use the MATLAB function `ode113` to solve this problem over the time interval $0 \le t \le 2$. Plot the true and computed solutions to make sure you've done this correctly.

   (d) Test the MATLAB solver by specifying different tolerances spanning several orders of magnitude. Create a table showing the maximum error in the computed solution for each tolerance and the number of function evaluations required to achieve this accuracy.

   (e) Repeat part (d) using the MATLAB function `ode45`, which uses an embedded pair of Runge-Kutta methods instead of Adams-Bashforth-Moulton methods.