Second-Order Ordinary Differential Equations (ODEs)

The Initial Value Problem

A second-order ODE is any equation involving a second derivative, but no higher derivative, of an unknown function. In its most general form, it has the appearance

$$F(x, y, y', y'') = 0, \tag{1}$$

in which $y(x)$ the function of interest and x is the independent variable.

A *solution* of equation (1) on an interval I is a function ϕ that satisfies the equation for all x in I. That is,

$$F(x, \phi(x), \phi'(x), \phi''(x)) = 0 \qquad \forall x \in I.$$

A *second-order initial value problem* has the form

$$F(x, y, y') = 0, \qquad y(x_0) = A, \qquad y'(x_0) = B, \tag{2}$$

in which x_0, A and B are given numbers. The conditions $y(x_0) = A$ and $y'(x_0) = B$ are called *initial conditions.*

Linear Second-Order ODEs

A second-order linear ODE has the form

$$R(x)y''(x) + P(x)y'(x) + Q(x)y(x) = S(x). \tag{3}$$

in which $R(x), P(x), Q(x)$ and $S(x)$ are continuous functions on some interval I. If $S(x) \equiv 0$, that is,

$$R(x)y''(x) + P(x)y'(x) + Q(x)y(x) = 0. \tag{4}$$

then the equation is called *homogeneous.*

Theorem *Let y_1 and y_2 be solutions of the homogeneous equation (4) on an interval I. Then any linear combination, $C_1y_1 + C_2y_2$, of these solution is also a solution.*

Definition Two functions f and g are linearly dependent on an open interval I if, for some constant C, either $f(x) = Cg(x)$ for all x in I. If f and g are not linearly dependent on I, then they linearly independent on the interval.

Theorem *Let y_1 and y_2 be solutions of the homogeneous equation (4) on an open interval I. Then,*

1. *Either the Wronskian, $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$, is equal to zero for all x in I, or $W(x) \neq 0$ for all x in I.*
2. *y_1 and y_2 are linearly independent on I if and only if $W(x) \neq 0$ for all x in I.*

Theorem *Let y_1 and y_2 be linearly independent solutions of the homogeneous equation (4) on an open interval I. Then, every solution of this ODE on I is a linear combination of y_1 and y_2.*

Definition Let y_1 and y_2 be solutions of the homogeneous equation (4) on an open interval I.

1. y_1 and y_2 form a fundamental set of solutions on I if y_1 and y_2 are linearly independent on I.
2. When y_1 and y_2 form a fundamental set of solutions, we call $C_1 y_1 + C_2 y_2$, with C_1 and C_2 arbitrary constants, the general solution of (4) on I.

Theorem *Let $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ be a general solution of the homogeneous equation (4) on an open interval I. Then, the general solution of equation (3) is given by*

$$y(x) = y_c(x) + y_p(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x),$$

where y_p is any solution of equation (3) on I.

Algorithm for Solving a Second-Order Linear ODE (3):

1. Find the general solution, $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$, of the associated homogeneous equation (4).
2. Find any solution, y_p of the equation (3) itself. There are two methods available:
   (a) *The Method of Undetermined Coefficients*
   (b) *The Method of Variation of Parameters*
3. Write the general solution $y(x) = y_c(x) + y_p(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$. This expression consists all possible solutions of equation (3) on the interval.

The Constant Coefficient Homogeneous Linear ODE: $ay'' + by' + cy = 0$

Find the roots, λ_1 and λ_2 of the characteristic equation $a\lambda^2 + b\lambda + c = 0$. Namely,

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. $\lambda_1 \neq \lambda_2$, real: $y_c(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
2. $\lambda_1 = \lambda_2 = \lambda$, real: $y_c(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$
3. $\lambda_{1,2} = \alpha \pm i\beta$: $y_c(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$

The Method of Variation of Parameters for Equation (3):

$$y_p(x) = u(x) y_1(x) + v(x) y_2(x),$$

where

1. y_1 and y_2 form a fundamental set of solutions of the associated homogeneous equation (4).
2. $u(x) = -\int \frac{y_2(x) S(x)}{R(x) W(x)}\, dx$, $v(x) = \int \frac{y_1(x) S(x)}{R(x) W(x)}\, dx$ and $W(x) = y_1(x) y_2'(x) - y_1'(x) y_2(x)$.

The Method of Undetermined Coefficients for Equation (3):

1. If $S(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$y_p(x) = x^s \left(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\right)$$

2. If $S(x) = e^{\alpha x}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)$, then

$$y_p(x) = x^s e^{\alpha x}\left(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\right)$$

3. If $S(x) = e^{\alpha x}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)\left\{\begin{array}{l}\cos\beta x\\ \sin\beta x\end{array}\right.$, then

$$\begin{aligned} y_p(x) &= x^s e^{\alpha x}\left(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\right)\cos\beta x \\ &\quad + x^s e^{\alpha x}\left(B_n x^n + B_{n-1}x^{n-1} + \cdots + B_1 x + B_0\right)\sin\beta x \end{aligned}$$

Here s is the smallest nonnegative integer ($s = 0, 1, 2$) which will ensure that no term in $y_p(x)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of characteristic equation, α is a root of characteristic equation, and $\alpha + i\beta$ is a root of characteristic equation, respectively.