

Sequences

1. If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.
2. **The Squeeze Theorem:** If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.
3. If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.
4. **Monotonic Sequence Theorem:** Every bounded, monotonic sequence is convergent.
5. The sequence r^n is convergent if $-1 < r < 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Series

1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.
2. **The Test for Divergence:** If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. **The Integral Test:** Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. in other words:
 - (a) If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
 - (b) If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.
4. **The Comparison Test:** Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
 - (a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum_{n=1}^{\infty} a_n$ is also convergent.
 - (b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum_{n=1}^{\infty} a_n$ is also divergent.

5. **The Alternating Series Test:** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

$$(a) \quad b_{n+1} < b_n \quad \text{for all } n$$

$$(b) \quad \lim_{n \rightarrow \infty} b_n = 0$$

then the series is convergent.

6. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

7. **The Ratio Test:**

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note • If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test gives no information.

8. The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

9. The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

10. For a given power series $\sum_{n=1}^{\infty} c_n(x - x_0)^n$ there are only three possibilities:

(a) There is a positive number R , called the **radius of convergence**, such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

(b) The series converges only when $x = a$ and the radius of convergence $R = 0$.

(c) The series converges for all x and the radius of convergence $R = \infty$.

The **interval of convergence** of a power series is the interval that consists of all the values of x for which the series converges.