Second-Order Linear Equations

A **second-order linear differential equation** has the form

$$P(x)\frac{d^2y}{dx} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \tag{1}$$

where P, Q, R, and G are continuous functions.

If $G(x) = 0$ for all x, Equation (1) is called **homogeneous** linear equation, that is, the form of a second-order homogeneous linear equation is

$$P(x)\frac{d^2y}{dx} + Q(x)\frac{dy}{dx} + R(x)y = 0 \tag{2}$$

If $G(x) \neq 0$ for some x, Equation (1) is **nonhomogeneous**.

A second-order nonhomogeneous linear differential equation with **constant coefficients** has the form

$$ay'' + by' + cy = G(x) \tag{3}$$

where a, b and c are constants, $a \neq 0$, and G is a continuous function.
The related homogeneous equation

$$ay'' + by' + cy = 0 \tag{4}$$

is called the **the complementary equation**.

Theorem *If y_1 and y_2 are linearly independent solutions of Equation (2) (Equation (4)), then the general solution is given by*

$$y_c(x) = C_1y_1(x) + C_2y_2(x)$$

where C_1 and C_2 are arbitrary constants. Two solutions are called linearly independent if neither y_1 nor y_2 is a constant multiple of the other.

Theorem *The general solution of the nonhomogeneous differential equation (1) (Equation (3)) can be written as*

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is the general solution of the complementary Equation (2) (Equation (4)) and $y_p(x)$ is a particular solution of Equation (1) (Equation (3)).

An **initial-value problem** for a second-order equation consists of finding a solution y of the differential equation that also satisfies initial conditions of the form: $y(x_0) = y_0, y'(x_0) = y_1$, where y_0 and y_1 are given constants.

A **boundary-value problem** for a second-order equation consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form: $y(x_0) = y_0, y(x_1) = y_1$, where y_0 and y_1 are given constants.

Solutions of $ay'' + by' + cy = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y_c(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y_c(x) = C_1 e^{rx} + C_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y_c(x) = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$

Two methods to find a particular solution $y_p(x)$:

- The Method of Undetermined Coefficients.
- The Method of Variation of Parameters.

Undetermined Coefficients

$G(x) =$	First try $y_p(x) =$
$C_n x^n + \cdots + C_1 x + C_0$	$A_n x^n + \cdots + A_1 x + A_0$
Ce^{kx}	Ae^{kx}
$C \cos kx + D \sin kx$	$A \cos kx + B \sin kx$
Modification: If any term of $y_p(x)$ is a solution of the complimentary equation, multiply y_p by x (or by x^2 if necessary).	

Variation of Parameters

We look for a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$ of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the complementary equation $ay'' + by' + cy = 0$. The functions $u_1(x)$ and $u_2(x)$ are solutions of the following system of equations:

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ a\,(u_1' y_1' + u_2' y_2') = G(x) \end{cases}$$

The method is called is variation of parameters because we have varied the parameters C_1 and $C_2$to make them functions.