Applications of Integration

Areas between Curves

- The area A of the region that lies under the curve $y = f(x)$, and the lines $x = a, x = b$, where f is continuous and $f(x) \geq 0$ for all x in $[a, b]$, is

$$A = \int_a^b f(x)\,dx$$

- The area A of the region that lies under the curve $y = f(x)$, and the lines $x = a, x = b$, where f is continuous and $f(x) \leq 0$ for all x in $[a, b]$, is

$$A = \Big| \int_a^b f(x)\,dx \Big|$$

- The area A of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a, x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\; dx$$

Volumes

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, trough x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \int_a^b A(x)\,dx$$

Arc Length

- If a smooth curve with parametric equations $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then its length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \equiv \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

- If a curve has the equation $y = f(x), a \leq x \leq b$, then the length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \equiv \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

- If a curve has the equation $x = f(y), a \leq y \leq b$, then the length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy \equiv \int_a^b \sqrt{(f'(y))^2 + 1}\,dy$$

Average Value of a Function

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$