The Definite Integral

Definition of a Definite Integral If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $a = x_0, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and we choose sample points $x_1^*, x_2^*, \ldots, x_n^*$ in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Remarks.
1. Because we have assumed that f is continuous, it can be proved that the limit in Definition always exists and gives the same value no matter how we choose the sample point x_i^*.
2.The symbol $\int$ was introduced by Leibnitz and is called an **integral sign**. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**: a is the **lower limit** and b is the **upper limit**. The procedure of calculating an integral is called **integration**.
3. The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Properties of the Definite Integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

1. $\int_a^b cdx = c(b - a)$, where c is any constant.

2. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant.

4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

5. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

6. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

7. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. $\dfrac{d}{dx} \int_a^x f(t)dt = f(x)$.

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Total Change Theorem The integral of a rate of change is the total change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Indefinite Integrals

The notation $\int f(x)dx$ is used for an antiderivative of and is called an **indefinite integral**. Thus

$$\int f(x)dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ *is a number, whereas an indefinite integral* $\int f(x)dx$ *is a function.*

Evaluation Theorem If f is continuous function on the interval $[a, b]$, then

$$\int_a^b f(x)dx = \int f(x)dx\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f, that is $F' = f$.

Table of Indefinite Integrals

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$$

$$\int cf(x)dx = c\int f(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -cscx + C$$

$$\int \frac{1}{x^2+1}dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin x + C$$