Second-Order Linear Equations

Example 1 (section 7.8, pg. 1172, problem 10). Solve the initial-value problem using the method of undetermined coefficients:

$$y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1,\ y'(0) = 0$$

Solution:
1) First, we solve the corresponding homogeneous equation $y''+y'-2y=0$. The characteristic equation is $r^2+r-2=(r-1)(r+2)=0$ with roots $r=1,-2$. So the general solution of the homogeneous equation is

$$y_c(x) = C_1e^x + C_2e^{-2x}$$

2) Second, we look for a particular solution of the given differential equation. Since $G(x) = x + \sin 2x$, we seek a particular solution of the form

$$y_p(x) = Ax + B + C\cos 2x + D\sin 2x$$

Then

$$y_p'(x) = A - 2C\sin 2x + 2D\cos 2x, \quad y_p'' = -4C\cos 2x - 4D\sin 2x$$

So, substituting into the given differential equation, we have

$$-4C\cos 2x - 4D\sin 2x + A - 2C\sin 2x + 2D\cos 2x - 2\left(Ax + B + C\cos 2x + D\sin 2x\right) = x + \sin 2x$$

or

$$-2Ax + (A-2B) + (-4C+2D-2C)\cos 2x + (-4D-2C-2D)\sin 2x = x + \sin 2x$$

or

$$-2Ax + (A-2B) + (-6C+2D)\cos 2x + (-6D-2C)\sin 2x = x + \sin 2x$$

This is true if

$$-2A = 1 \qquad A - 2B = 0 \qquad -6C + 2D = 0 \qquad -6D - 2C = 1$$

The solution of the system is

$$A = -\frac{1}{2} \qquad B = -\frac{1}{4} \qquad C = -\frac{1}{20} \qquad D = -\frac{3}{20}$$

A particular solution is therefore

$$y_p(x) = -\frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$$

3) By the superposition principle, the general solution is

$$y(x) = y_c(x) + y_p(x) = C_1e^x + C_2e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$$

The final step is to find C_1 and C_2 such that the general solution $y(x)$ satisfies the initial conditions $y(0) = 1, y'(0) = 0$. Imposing the initial condition $y(0) = 1$, we get

$$y(0) = C_1 + C_2 - \frac{1}{4} - \frac{1}{20} = 1$$

or

$$C_1 + C_2 = \frac{13}{10}$$

To impose the other initial condition we first differentiate the solution:

$$y'(x) = C_1 e^x - 2C_2 e^{-2x} - \frac{1}{2} + \frac{2}{20}\sin 2x - \frac{6}{20}\cos 2x$$

So

$$y'(0) = C_1 - 2C_2 - \frac{1}{2} - \frac{6}{20} = 0$$

or

$$C_1 - 2C_2 = \frac{4}{5}$$

The solution of the system

$$C_1 + C_2 = \frac{13}{10} \qquad C_1 - 2C_2 = \frac{4}{5}$$

is

$$C_1 = \frac{3}{5} \qquad C_2 = \frac{7}{10}$$

Therefore

$$y(x) = \frac{3}{5}e^x + \frac{7}{10}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$$

Example 2 (section 7.8, pg. 1172, problem 26). Solve the initial-value problem using the method of variation of parameters:

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$$

1) First, we solve the corresponding homogeneous equation $y'' + 4y' + 4y = 0$. The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$ with roots $r_{1,2} = -2$. So the general solution of the homogeneous equation is

$$y_c(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

2) Second, we look for a particular solution of the given differential equation. Using variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x)e^{-2x} + u_2(x)xe^{-2x}$$

Then

$$\begin{cases} u_1'e^{-2x} + u_2'xe^{-2x} = 0 \\ u_1'(e^{-2x})' + u_2'(xe^{-2x})' = \dfrac{e^{-2x}}{x^3} \end{cases} \qquad \text{or} \qquad \begin{cases} u_1'e^{-2x} + u_2'xe^{-2x} = 0 \\ u_1'(-2e^{-2x}) + u_2'(e^{-2x} - 2xe^{-2x}) = \dfrac{e^{-2x}}{x^3} \end{cases}$$

Solving the above system, we get

$$u_1' = -\frac{1}{x^2} \qquad u_2' = \frac{1}{x^3}$$

or

$$u_1(x) = \frac{1}{x} \qquad u_2(x) = -\frac{1}{2x^2}$$

Therefore

$$y_p(x) = u_1(x)e^{-2x} + u_2(x)xe^{-2x} = \frac{1}{x}e^{-2x} - \frac{1}{2x^2}xe^{-2x} = \frac{1}{2x}e^{-2x}$$

3) By the superposition principle, the general solution is

$$y(x) = y_c(x) + y_p(x) = C_1e^{-2x} + C_2xe^{-2x} + \frac{1}{2x}e^{-2x}$$