ECG790C - Problem Set 5 - Spring 2009

1. Consider how to obtain an approximation of $\frac{\partial f(x,y)}{\partial x \partial y}$ using a linear combination of the points $f(x,y)$, $f(x+\delta,y)$ and $f(x,y+\delta)$, $f(x+\delta,y+\delta)$. Specifically show that

$$\frac{\partial f(x,y)}{\partial x \partial y} = \frac{af(x,y) + bf(x+\delta,y) + cf(x,y+\delta) + df(x+\delta,y+\delta)}{\delta^2} + O(\delta^h)$$

and in doing so determine the values of a, b, c, d and h. It might be of use to note that the Taylor expansion of a function of two variables is

$$f(x+\delta, y+\eta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\partial^{i+j} f(x,y)}{\partial^i x \partial^j y} \frac{\delta^i \eta^j}{i!j!}$$

Write a MATLAB function that computes this approximation and test it using the function $\exp(xy)$. Compute and plot the error of the approximation for alternative values of δ and use this to draw a conclusion about the optimal value of δ.

2. This problem requires that you use one of MATLAB's ODE solvers. Consider the system in which, for $1 < i < n$

$$\frac{dV_i}{d\tau} = a_- V_{i-1} + a_0 V_i + a_+ V_{i+1}$$

where

$$a_- = \frac{\sigma^2}{2\delta^2} - \frac{2r - \sigma^2}{4\delta}$$

$$a_0 = -\frac{\sigma^2}{\delta^2} - r$$

$$a_+ = \frac{\sigma^2}{2\delta^2} + \frac{2r - \sigma^2}{4\delta}$$

dV_1 and dV_n are handled differently.

$$\frac{dV_1}{d\tau} = a_1 V_1 + a_2 V_2 + a_3 V_3 + a_4 V_3$$

where

$$a_1 = \frac{\sigma^2}{\delta^2} - \frac{6r - 3\sigma^2}{4\delta} - r$$

$$a_2 = -\frac{5\sigma^2}{2\delta^2} + \frac{2r - \sigma^2}{\delta}$$

$$a_3 = \frac{2\sigma^2}{\delta^2} - \frac{2r - \sigma^2}{4\delta}$$

$$a_4 = -\frac{\sigma^2}{2\delta^2}$$

$$\frac{dV_n}{d\tau} = a_n V_n + a_{n-1} V_{n-1} + a_{n-2} V_{n-2} + a_{n-3} V_{n-3}$$

where

$$a_n = \frac{\sigma^2}{\delta^2} + \frac{6r - 3\sigma^2}{4\delta} - r$$

$$a_{n-1} = -\frac{5\sigma^2}{2\delta^2} - \frac{2r - \sigma^2}{\delta}$$

$$a_{n-2} = \frac{2\sigma^2}{\delta^2} + \frac{2r - \sigma^2}{4\delta}$$

$$a_{n-3} = -\frac{\sigma^2}{2\delta^2}$$

Note that we can express this most succinctly as $dV/d\tau = FV$ where F is an $n \times n$ matrix. Except for the first and last rows, F is tridiagonal meaning it has non-zero elements only on the diagonal and the first sub- and super-diagonals. This is most easily defined using `spdiags`. F should definitely be defined as a sparse matrix so that unnecessary arithmetic in computing FV is avoided.

To use Matlab's ODE solvers (I suggest either `ode45` or `ode15s`) you will need to implement a function that is passed t, V and any additional parameters you define (e.g., F) and returns dV/dt. In this case V and dV/dt are both n-vectors.

Let s_i be n evenly spaced points on $[\ln(K/u), \ln(Ku)]$ and define $S_i = \exp(s_i)$. Initialize $V_i(0) = \max(K - S_i, 0)$. Solve the ODE over in interval $t \in [0, T]$.

Create a plot of the points (S_i, V_i) and compare your answer to the points obtained using the Black-Scholes formula for a put option on a stock with no dividends (with stock price S_i, strike price K, interest rate r, volatility σ and time-to-maturity T).

Use the following parameters $r = 0.05$, $\sigma = 0.25$, $K = 1$, $T = 1$, $u = 3$, and $n = 101$.

Note this problem asks you to implement a method for solving derivative pricing problems that we will discuss later in the course (you don't need to know why this works to get it to work!).

3. Use Gaussian quadrature and the trapezoid method to compute $E[|X|]$ and $E[\exp(X)]$ where $X \sim N(0,1)$ (the PDF is $\exp(-x^2/2)/\sqrt{2\pi}$). Explore how the approximations change with the number of nodes used and draw some conclusions. Note that the exact answers are $\sqrt{2/\pi}$ and $\exp(1/2)$.