ECG790C - Take-home Final - Spring 2009

This exam is due either one week after I send it to you or Sunday, May 3 at 6:00 PM. You should not discuss this exam with anyone except me. All work should be your own. If you have difficulty understanding what is being asked in a problem, please contact me by email.

1. The c-probability Value-at-Risk (VaR) of an asset or portfolio over a holding period of h is defined as

$$Prob(V_{t+h} < \text{VaR}(c,h)|I_t) \leq c$$

where V_t is the portfolio value at time t and I_t denotes information available at time t.

For example, suppose that the price of an asset is described by

$$dS = \mu S dt + \sigma S dW$$

The h period ahead transition probability is lognormal; specifically

$$\ln(S_{t+h})|S_t \sim N\Big(\ln(S_t) + (\mu - \sigma^2/2)h, \sigma^2 h\Big)$$

Thus

$$\ln(\text{VaR}(c,h)) = \Phi^{-1}(c)\sigma\sqrt{h} + \ln(S_t) + (\mu - \sigma^2/2)h$$

where Φ is the standard normal CDF.

Under the Basel II accord, VaR measures were adopted as a standard to set banks capital requirements (in the light of present circumstances it is questionable how effective this was).

This problem asks you to perform a VaR analysis on a portfolio. To keep this relatively simple, suppose that there are n stocks in this portfolio and n_i derivatives on the ith stock. The portfolio itself consist of w_i units of the ith stock and w_{ij} units of the jth derivative on the ith stock. For example, if there are $n = 2$ stocks, $n_1 = 1$ derivative on stock 1 and $n_2 = 2$ derivatives on stock 2 and you hold $w_1 = 3$ units of stock 1, $w_2 = 2$ units of stock 2, $w_{11} = 1$ unit of the derivative on stock 1, $w_{21} = 4$ units of the first derivative on stock 2 and $w_{22} = 5$ units of the second derivative on stock 2, the current value of this portfolio is

$$V = 3V_1 + 2V_2 + V_{11} + 4V_{21} + 5V_{22} + C$$

where C is the value of cash (actually this is held as risk free bonds that appreciate at rate r).

In order to determine the VaR of this portfolio, we need to compute the h period ahead probability distribution of the value of the portfolio. To keep things simple, it is assumed that the portfolio will not change in composition over the next h periods except, perhaps, that some of the derivatives will mature (if this occurs, assume that any proceeds or payments are added or subtracted from the cash account C).

With these assumptions the distribution of V is determined by the joint distribution the two stock prices (again for simplicity assume that the derivatives depend only on the price of the underlying stock as would be the case for simple European options).

Suppose that the price of the ith stock is described by

$$dS_i = rS_i dt + \sigma_i S_i dW_i$$

and that the Brownian motions are correlated with correlation matrix R. This means that the increments of the Brownian motions can be obtained using `randn(1,m)*chol(R)*sqrt(Delta)`. It also means that European options written on the stocks can be obtained using the Black-Scholes formula.

Consider a portfolio with $n = 3$. The details of the portfolio that is held is as follows.

Stock	σ_i
1	0.4
2	0.45
3	0.35

The correlation matrix for the Brownian motions is

$$R = \begin{bmatrix} 1 & 0.65 & 0.7 \\ 0.65 & 1 & 0.75 \\ 0.7 & 0.75 & 1 \end{bmatrix}$$

The current values of the stocks are 820, 605 and 710, respectively and the cash account beings with $C = 5000$.

The details of the derivatives held are (here K is the exercise or strike price, T is the exercise date, long means it was purchased and short means it was sold; all of these options are European):

Stock	Type	K	T	#
1	Call	800	1.5	-2
1	Call	850	1	2
2	Put	600	0.5	1
3	Put	725	0.75	1
3	Put	700	1	-2

(here a negative number of a derivative refers to a short or selling position). The risk free interest rate $r = 0.05$. All of the stocks are non-dividend paying.

Determine the VaR for c equal to 0.1, 0.05 and 0.01 for $h \in [0, 1]$ and plot it. Also create a table of values for these levels of c for the one week, one month and one year VaR.

Your code should be written so it can be easily altered to include more stocks and/or options and alternative parameter values.

2. Suppose a firm can operate a machine at any level $x \in [0, 1]$ (think of x as the fraction of operating capacity). Operating the machine, however, diminishes the machine's productivity S, which also has a random component

   $$dS = -xSdt + \sigma SdW$$

   The machine produces xS units of a good that sells for price p and it costs cx to operate (so the reward function is $(ps - c)x$. Future profits are discounted at rate r.

   State the Hamilton-Jacobi-Bellman equation for this problem. State the optimal decision rule. With $p = 1$, $c = 0.5$, $\sigma = 0.05$, and $r = 0.08$, use `scsolve` (or write your own code) to approximate the value of the machine V. It is optimal not to operate the machine anytime $S < S^*$; determine S^*.

   Suppose that x can take on only the values 0 or 1. In this case the problem can be viewed as a regime switching problem. Use `rssolve` (or write your own code) to solve the problem and compare the solutions to the two problems.

3. As you know, the Euler approximation of the SDE

$$dS = \mu(S)dt + \sigma(S)dW$$

uses

$$S_{t+\Delta} \approx S_t + \mu(S_t)\Delta + \sigma(S_t)\sqrt{\Delta}z$$

where z is a standard Normal random variable.

A first order Milstein approximation adds the term

$$\frac{1}{2}\sigma(S_t)\sigma'(S_t)\Delta(z^2 - 1)$$

(using the same z).

Demonstrate that approximation errors in paths generated using the Milstein approximation declines at rate Δ whereas errors from Euler's method (generally) decline at rate $\sqrt{\Delta}$. Specifically, determine the rate at which the norm of the differences between the actual path and the approximate path increases as Δ increases. To do this you will need to know the "true" path; this can be well approximated by using very small time steps and then using the same Brownian paths sampled at higher values of Δ.

Note that for processes for which $\sigma'(S) = 0$, i.e., for processes with constant variance, the Euler and first order Milstein methods are identical. In this case Euler also has first order accuracy.

Demonstrate these statements using the following two processes (the first having non-constant variance)

$$dS = \lambda S(\mu - S)dt + \sigma S dW$$

and

$$dS = (\alpha/S - \beta S)dt + \eta dw$$

with the following parameter values: $\lambda = 0.5$, $\mu = 1.5$, $\sigma = 0.2$, $\alpha = 2$, $\beta = 1$ and $\eta = 0.75$. Start both processes at 1.6.

4. A callable bond can be terminated (called) by the seller, at which point the seller pays the bond holder the face value of the bond (if there are discrete dividends, they also pay the accumulated dividend a the time the bond is called).

   Compute the value of a 5 year bond paying a continuous coupon with rate δ when the risk-free instantaneous interest rate is described by

$$dr = \kappa(\mu - r)dt + \sigma r dW$$

   Use the parameter values $\kappa = .2$, $\alpha = .05$, $\sigma = 0.6$ and $\delta = 0.025$.

   Plot the time 0 value of the bond with and without the callable feature. Intuitively, a bond seller would desire to exercise the option to call the bond when the interest rate is very low. Plot the relationship between the time-to-maturity and the interest rate below which the bond would be called and create a table of these values for 1, 2, 3 and 4 years to maturity.